P–3 Mathematics

CA Preschool Curriculum Framework

CA Common Core Mathematical Practices

Progression Documents for Common Core

Fourth Edition

SAN MATEO COUNTY OFFICE OF EDUCATION
Anne E. Campbell • County Superintendent of Schools

Includes excerpts from (1) California Preschool Curriculum Framework, Volume 1 (2010), (2) Common Core State Standards Initiative, Common Core State Standards for Mathematics, (3) Standards for Mathematical Practice Professional Learning Module, California Department of Education, 2012 (4) Curriculum and Instruction Steering Committee Common Core State Standards Tooklit and (5) Progression Documents for the Common Core Math Standards from the University of Arizona

HEISING-SIMONS FOUNDATION

Preschool Learning Foundations and Common Core State Standards, Mathematics
Strand and Domain Alignment

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Mathematical Reasoning	Standards for Mathematical Practice (MP)

Table 1
Overview of the Alignment of the California Preschool Learning Foundations with Key Early Education Resources

Domains					
California Preschool Learning Foundations	**California Infant/Toddler Learning and Development Foundations**	**California Kindergarten Content Standards**	**Common Core State Standards**	**Head Start Child Development and Early Learning Framework**	**Additional Domains in the Head Start Child Development and Early Learning Framework with Corresponding Content**
Social–Emotional Development	Social–Emotional Development	Health Education Mental, Emotional, and Social Health		Social & Emotional Development	Approaches to Learning Logic & Reasoning
Language and Literacy	Language Development	English–Language Arts	English–Language Arts	Language Development Literacy Knowledge & Skills	
English—-Language Development	Language Development	English–Language Development		English Language Development	Literacy Knowledge & Skills
Mathematics	Cognitive Development	Mathematics	Mathematics	Mathematics Knowledge & Skills	Logic & Reasoning Approaches to Learning
Visual and Performing Arts	All Domains	Visual and Performing Arts		Creative Arts Expression	Logic & Reasoning
Physical Development	Perceptual and Motor Development Cognitive Development	Physical Education		Physical Development & Health	
Health	All Domains	Health Education		Physical Development & Health	
History–Social Science	Social–Emotional Development Cognitive Development	History–Social Science		Social Studies Knowledge & Skills	Social & Emotional Development
Science	Cognitive Development Language Development	Science		Science Knowledge & Skills	Approaches to Learning Logic & Reasoning

California Preschool Curriculum Framework

Volume 1

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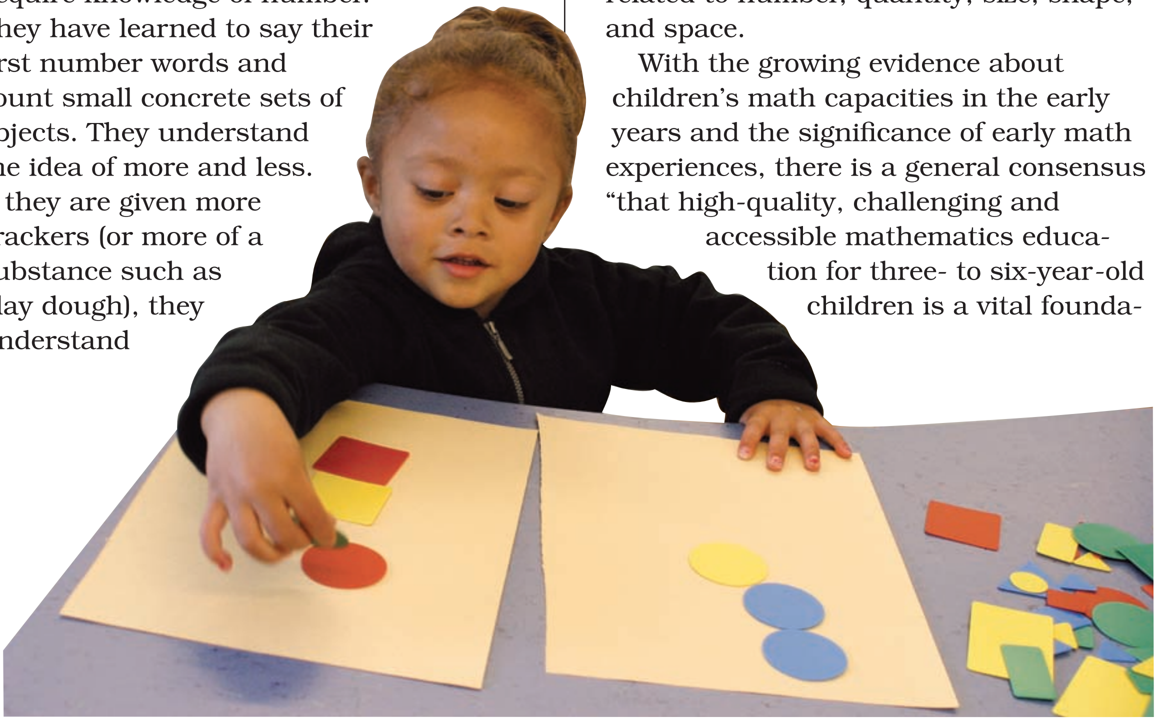
CHAPTER 6

Mathematics

Mathematics is a natural part of the preschool environment. Young children actively construct mathematical knowledge through everyday interactions with their environment, whether inside or outside. When building in the block area or sorting blocks by shape, children explore geometry in the real world. When measuring two cups of flour and three spoons of sugar in a cooking activity, they learn principles of measurement. Climbing in and out of cardboard boxes, crawling through a tunnel, or riding a bike helps children develop a sense of spatial relationships (e.g., on, under, over). Mathematics learning grows naturally from children's curiosity and enthusiasm to learn and explore their environment. Teachers should encourage children's natural enthusiasm and interest in doing mathematics and use it as a vehicle for supporting the development of children's mathematical concepts and skills.

Young children seem to have an innate sense of informal mathematics. They develop a substantive body of informal knowledge of mathematics from infancy throughout the preschool years. By the age of three, they have already begun to acquire knowledge of number. They have learned to say their first number words and count small concrete sets of objects. They understand the idea of more and less. If they are given more crackers (or more of a substance such as play dough), they understand they have *more* than they did before, and if some were taken away, they now have *less.* During the preschool years, children continue to show a spontaneous interest in mathematics and further develop their mathematical knowledge and skills related to number, quantity, size, shape, and space.

With the growing evidence about children's math capacities in the early years and the significance of early math experiences, there is a general consensus "that high-quality, challenging and accessible mathematics education for three- to six-year-old children is a vital founda-



tion for future mathematics learning."[1] High-quality mathematics education in preschool is not about elementary arithmetic being pushed down onto younger children. It is broader than mere practice in counting and arithmetic. It is about children experiencing mathematics as they explore ideas of more and less, count objects, make comparisons, create patterns, sort and measure objects, and explore shapes in space. Mathematics learning happens throughout the day, and it is integrated with learning and developing in other developmental domains such as language and literacy, social-emotional, science, music, and movement.

Teachers have a significant role in facilitating children's construction of mathematical concepts. They may not always realize the extent to which their current everyday classroom practices support children's mathematical development. For example, when singing with children "Five Little Ducks Went Out One Day," incorporating finger play with counting, the teacher develops children's counting skills and understanding of number. Discussing with children how many children came to school today and how many are missing supports children's arithmetic and reasoning with numbers. Playing with children in the sandbox by filling up different cups with sand and discussing which cup is the *smallest* or the *largest* or how many cups of sand it would take to fill up a bucket introduces children to concepts of comparison and measurement. Preschool teachers nurture children's natural enthusiasm and interest in learning mathematics. They help children build their knowledge and skills of mathematics over time, by providing a mathematically rich environment, by modeling mathematical thinking and reasoning, and by introducing children to the language of math.[2] Teachers guide, support, and challenge children in the journey of exploring and constructing mathematical knowledge. As stated by the National Council of Teachers of Mathematics (NCTM):

> . . . adults can foster children's mathematical development by providing environments rich in language, where thinking is encouraged, uniqueness is valued, and exploration is supported. Play is children's work. Adults support young children's diligence and mathematical development when they direct attention to the mathematics children use in their play, challenge them to solve problems and encourage their persistence.[3]

When teachers join children in becoming keen observers of their environment and in reasoning about numbers, shapes, and patterns, mathematics is enjoyable and exciting for all.

Guiding Principles

The following principles will guide teachers' classroom practices in establishing a high-quality, challenging, and sensitive early mathematics preschool program. These principles are partially based on the ten recommendations in *Early Childhood Mathematics:*

Promoting Good Beginnings set forth by the National Association for the Education of Young Children and NCTM in 2002.

- **Build on preschool children's natural interest in mathematics and their intuitive and informal mathematical knowledge**
  Young children are mathematically competent, motivated, and naturally interested in exploring mathematical ideas and concepts. Teachers should recognize children's early mathematical competence and build on children's disposition to use mathematics as a way to make sense of their world.

- **Encourage inquiry and exploration to foster problem solving and mathematical reasoning**
  Mathematical reasoning and problem solving are natural to all children as they explore the world around them. The most powerful mathematics learning for preschool children often results from their own explorations. Teachers should maintain an environment that nurtures children's inquiry and exploration of mathematical ideas and that values problem solving. They should ask children questions to stimulate mathematical conversations and encourage mathematical reasoning through everyday interactions. Teachers' meaningful questions can lead to clarifications, more advanced challenges, and the development of new understandings.

- **Use everyday activities as natural vehicles for developing preschool children's mathematical knowledge**
  Children can learn mathematical concepts through play and everyday activities as they interact with materials and investigate problems. Putting toys away, playing with blocks, helping to set the table before snack, or playing with buckets of varying sizes in the sand are all opportunities for children to learn about key mathematical concepts such as sorting, geometry, number, and measurements. Teachers should build upon the naturally occurring mathematics in children's daily activities and capitalize on "teachable moments" during such activities to extend children's mathematical understanding and interest.

- **Introduce mathematical concepts through intentionally planned experiences**
  In addition to the meaningful mathematics that preschool children acquire spontaneously through play and everyday activities, teachers should provide carefully planned experiences that focus children's attention on particular mathematical concepts, methods, and the language of math. Mathematical experiences planned in advance would allow teachers to present concepts in a logical sequence and forge links between previously encountered mathematical ideas and new applications. Teachers should build on what the child already knows and reasonably challenge the child in acquiring new skills or knowledge. Teachers can foster children's understanding

of mathematical concepts over time through intentional involvement with mathematical ideas in preschool and by helping families extend and develop these ideas.

- **Provide a mathematically rich environment**
Arranging a high-quality physical environment is important for children's mathematical development. It should offer children opportunities to experiment and learn about key mathematical concepts naturally throughout the classroom and throughout the day.

- **Provide an environment rich in language, and introduce preschool children to the language of mathematics**
Language is a critical element in mathematics. Children should be introduced to mathematical vocabulary as well as to natural language in meaningful contexts. During the preschool years, children learn mathematical language such as the number words, the names for shapes, words to compare quantity (e.g., *bigger, smaller),* and words to describe position and direction in space (e.g., *in, on, above).* Children often have an intuitive understanding of mathematical concepts but lack the vocabulary and the conceptual framework of mathematics. By introducing children to mathematical vocabulary, teachers help "mathematize" what children intuitively grasp. Language allows children to become aware of their mathematical thinking and to express it in words. Children with delays in development, especially in language development, may need more frequent repetition of the words combined with a demonstration of the concept.

- **Support English learners in developing mathematical knowledge as they concurrently acquire English**
Teachers should be aware of the challenges faced by children who are English learners and apply specific instructional strategies to help children learning English acquire mathematical concepts and skills. To provide children who are English learners with comprehensible information, they should simplify the terms they use, make extensive use of manipulatives, illustrate the meaning of words by acting and modeling whenever possible, and encourage children to use terms in their home language. Repetition, paraphrasing, and elaboration by the teacher also help preschool children who are English learners understand the content of the conversation. Teachers are encouraged to use mathematical terms as often as possible and in as many different settings as possible. Teachers' attentive and modified talk helps young children learning English to understand mathematical concepts and to develop the language skills they need to communicate mathematical ideas.[4]

- **Observe preschool children and listen to them**
Observe children thoughtfully, listen carefully to their ideas, and talk with them. Close observation allows teachers to identify thought-provoking moments through everyday play, where mathematical concepts can be clarified, extended, and reinforced, and children can be prompted to make new discoveries. Observing and listening to children also allows teachers to learn about children's interests and attitudes and to assess children's mathematical knowledge and skills.

Take into account that mathematical knowledge is not always expressed verbally. Children may know a lot about number, size, or shape without having the words to describe what they know.

Recognize and support the individual

Provide an environment in which *all* children can learn mathematics, set appropriately high expectations for all children, and support individual growth. Children differ in their strengths, interests, approaches to learning, knowledge, and skills. They may also have special learning needs. Young children, therefore, may construct mathematical understanding in different ways, at varying rates, and with different materials. To be effective, teachers should respond to each child individually. They should find out what young children already know and build on the children's individual strengths and ways of learning. Teachers should provide children with a variety of materials, teaching strategies, and methods to meet children's different learning styles and promote access to and attainment of mathematical concepts by all children. The strategies presented in the next sections for supporting children's development in the mathematics domain apply to all children. Children with disabilities and other special needs, like all children, benefit from multiple opportunities to experience math concepts through playful activities that build on their interests. They particularly benefit from hands-on activities, using a variety of manipulatives, and from teachers' support and verbal descriptions of what they are doing. If children are receiving special education services, teachers should ask for ideas from the specialists and families.

Establish a partnership with parents and other caregivers in supporting children's learning of mathematics

Parents and other caregivers should be partners in the process of supporting children's mathematics development. Parents serve as role models for children. When parents become involved in their children's mathematics education, children become more engaged and excited. Teachers should communicate to parents what preschool mathematics is about, age-appropriate expectations for mathematics learning at the preschool level, and how mathematics learning is supported in the preschool environment. They should also convey to parents the importance of mathematics and what they can do at home for supporting children's math development. By talking with parents, teachers could also learn about children's interests, natural knowledge, and home experiences related to math. They may need to remind parents about the numerous opportunities to talk with children about number, shape, size, and quantity during everyday home routines and activities. For example, while walking to school or taking the bus, parents can point out the yield signs, stop signs, and so on and say the name of each shape (triangle, rectangle, square) and can count the number of footsteps to the front door. While cooking, they can count the number of cups of rice or beans. Throughout the year, teachers should also provide parents with information about the child's development and progress in learning math concepts and skills.

Environments and Materials

Young children actively construct mathematical knowledge through everyday interactions with their environment. Setting up a high-quality physical environment is essential for children's mathematical development. The preschool environment sets the stage for children's physical and social exploration and construction of mathematical concepts. It should provide access to objects and materials that encourage children to experiment and learn about key mathematical concepts through everyday play.

- **Enrich the environment with objects and materials that promote mathematical growth.**
  Provide children with access to developmentally appropriate, challenging, and engaging materials. A high-quality environment offers children opportunities to count objects; to explore and compare objects' size, shape, weight, and other attributes; to measure; to sort and classify; and to discover and create patterns. For example, wooden blocks, geometric foam blocks, cylinders, cones, and boxes would encourage creativity while stimulating concepts of geometry. Collections of small items such as rocks, beads, cubes, buttons, commercial counters, and other items can be used for counting, sorting, and categorizing. Containers of different sizes and measuring cups and spoons can illustrate the concepts of volume and capacity. The environment should also include number-related books; felt pieces or finger puppets to go with the books; and counting games using dice, spinners, and cards. It may also include computer software and other technology materials focused on math. Materials and props will support all children in learning mathematics and are particularly important in teaching preschool children who are English learners. The props and materials give concrete meaning to the words children hear in the context of doing mathematics.

  Children with physical disabilities may need assistance in exploring the environment and manipulating objects. Children with motor impairments may explore through observation or may need assistance from an adult or a peer in manipulating objects to do things such as count, sort, compare, order, measure, create patterns, or solve problems. A child might also use adaptive materials (e.g., large manipulatives that are easy to grasp). Alternately, a child might demonstrate knowledge in these areas without directly manipulating objects. For example, a child might direct a peer or teacher to place several objects in order from smallest to largest. Children with visual impairments might be offered materials for counting, sorting, or problem solving that are easily distinguishable by touch. Their engagement is also facilitated by the use of containers, trays, and so forth of materials that clearly define their workspace.

- **Integrate math-related materials into all interest areas in the classroom.**
  Math naturally takes place throughout the classroom and throughout the day. Children explore objects and learn about shapes and numbers as they go about their daily routine and play in different areas in the classroom. Number symbols, for example, naturally appear throughout the classroom,

from real-life objects such as a tape measure, a telephone, a calculator or a scale to puzzles, stickers, books, and cards with numbers. Some teachers may choose to have a math table or a math area in the classroom for math-related materials, games, books, and manipulatives. In addition, the teacher should integrate math-related materials and props into all activity areas in the classroom. The dramatic play area can include a scale, a calculator, a measuring tape, and other math-related tools. The art area can include shape and number stickers, magazine cutouts of numbers, and shapes for collage making. The same tool can be used in various places throughout the environment. Measuring cups and spoons, for example, can be used for cooking, but also in the science or discovery area, in the dramatic play area, and for playing with sand and water.

- **Provide real-life settings in the preschool environment.**
Real-life settings to investigate, such as a grocery store, a restaurant, a woodshop, or a bakery, help children learn naturally about everyday mathematics. They present children with numerous opportunities for mathematical reasoning and problem solving. Such settings demonstrate for children mathematical concepts through props and concrete objects, familiarize children with numbers in their everyday use (e.g., price tags, labels, measurements) and with the function of various tools (e.g., a scale, a register, a measuring tape). A real-life setting such as a grocery store or bakery, for example, can engage children in sorting and classification of items, in measurement experiences (e.g., measuring the weight of produce), and in solving simple addition and subtraction problems. Children enjoy learning mathematics through the acting out of different roles in real-life settings.

- **Use materials and objects that are relevant and meaningful to the children in your group.**
Mathematical concepts and skills such as counting, sorting, and measuring can be learned with different materials and in various contexts. It is valuable to introduce math in a context that is familiar and relevant to children's life experiences. Use materials, books, and real-life settings that reflect the culture, ways of life, and languages of the children in the group. When mathematical concepts are embedded in a context that is personally relevant to individual children, experiences are more pleasurable and meaningful.

- **Use children's books to explore mathematics with children.**
Include books with mathematical content, and use children's literature to develop mathematical concepts. Children's books provide interesting and powerful ways to explore mathematics. Teachers can use books to introduce and illustrate different

mathematical concepts, to encourage the use of mathematical language, and to develop mathematical thinking. Some books, such as counting books and shape books, directly illustrate mathematical concepts. Other books, such as storybooks, provide context for mathematical reasoning (e.g., *The Very Hungry Caterpillar* or *Goldilocks and the Three Bears*). The following sections include suggestions about how teachers can use literature to present and discuss different mathematical concepts, including counting, addition and subtraction, patterns, shapes, comparison language, and spatial positions. Many stories can be acted out by including concrete objects and manipulatives. While reading aloud books with mathematical content, teachers can pose questions to children, ask them to predict what comes next based on an underlying principle or a repeated pattern in the story, or invite children to re-create stories in their own way. See the "Teacher Resources" on page 297 for a list of children's books with mathematical content and other related resources on the use of literacy in teaching mathematics. For ideas on adapting books for children with physical disabilities, please refer to the Literacy section on pages 106 and 107.

Be intentional and mindful in setting up and using the physical environment.
A math-rich environment is very important, but it does not guarantee that children will engage in meaningful mathematical experiences. The teacher should be intentional when planning a math-rich environment and think about how different math-related objects in the classroom can be utilized to promote meaningful mathematical exploration and reasoning. Teachers should allow children the time to become involved with the materials, help children reflect on what they are doing, and extend their learning and discoveries through questioning and mental challenges. The next sections include more detailed information about how to set up a rich physical environment to promote number sense, classification, measurement, and geometry concepts for all children.

Summary of the Strands and Substrands

The California preschool learning foundations in mathematics identify a set of age-appropriate goals expected for children at around 48 and 60 months of age in five developmental strands.

- **The Number Sense strand** refers to concepts of numbers and their relationships. It includes the development of counting skills, the understanding of quantities, recognizing ordering relations (which has more, fewer, or less), part-whole relationships, and a basic understanding of "adding to" and "taking away" operations.
- **The Algebra and Functions (Classification and Patterning) strand** concerns the development of algebraic thinking and reasoning. Included in this strand is the ability to sort, group, and classify objects by some attribute and to recognize, extend, and create patterns.
- **The Measurement strand** involves comparing, ordering, and measuring things. Included in this strand is the child's ability to compare and order objects by length, height, weight, or capacity; to use comparison vocabulary; and to begin to measure.

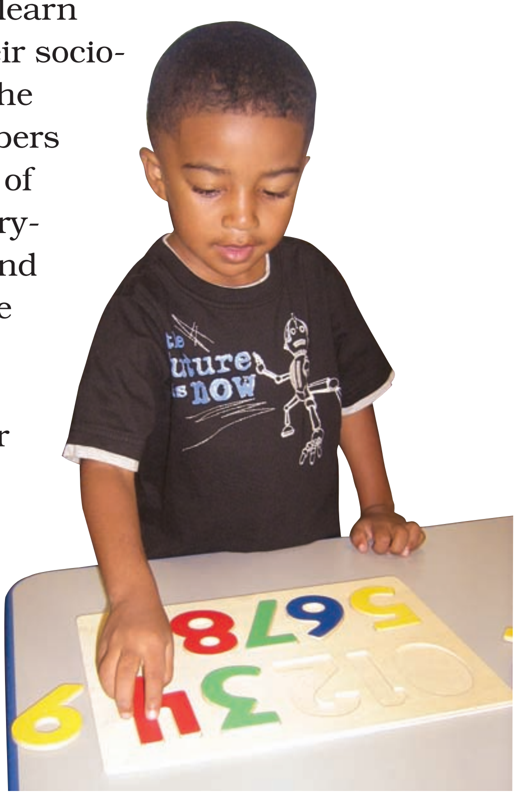
- **The Geometry strand** concerns the study of shapes and spatial relationships. Included in this strand is the child's ability to identify, describe and construct different shapes, and to identify and label positions in space.
- **The Mathematical Reasoning strand** is a process in learning and developing mathematical knowledge in all areas of mathematics. Included in this strand is the child's ability to reason and apply mathematical knowledge and skills to solve problems in the everyday environment.

Please refer to the map of the mathematics foundations on page 296 for a visual explanation of the terminology used in the preschool learning foundations.

The following curriculum framework in mathematics provides teachers with strategies to promote preschool children's reasoning and understanding of key mathematical concepts in each of the five strands. The strategies provide teachers with tools for building children's understanding of mathematics over time, through a mathematically rich environment, through interactions and conversations with children during play and everyday routines, and through intentionally planned mathematical experiences. Examples of "Mathematical Reasoning in Action" are interwoven throughout the chapters, illustrating children's reasoning about different mathematical concepts, whether in natural situations or while engaged in planned mathematical activities.

Number Sense

Number sense refers to children's concept of numbers and their relationships. It starts early on with an infant's ability to visually recognize the number of elements in a small set and continues with children's verbal counting, as they further develop the sense of quantity, number relationships (e.g., less than, greater than), and the fundamental understanding of addition and subtraction. Children enter preschool with an intuitive understanding of number and **operations** and with a natural curiosity and eagerness to learn about numbers. All children, whatever their socio-economic or cultural backgrounds, have the tendency to count and reason about numbers in everyday life. Children's intuitive sense of number does not imply, however, that everything they need to learn about numbers and operations comes naturally. Teachers have an extremely important role in supporting children's understanding of number and operations, making them aware of number concepts and introducing them to the language of mathematics. All preschool children benefit from opportunities throughout the day to count, compare quantities, and solve problems involving numbers. The following strategies provide suggestions as to how teachers can help children build number sense.



1.0 Understanding Number and Quantity

From a very young age, children can determine the quantity of objects in a small set without counting **(subitizing)** and can label "two" or "three" when looking at small collections of objects. Repeated counting experiences develop a child's counting skills and her understanding of quantity. Children learn that counting determines the quantity of objects in a set (e.g., "One, two, three, four, there are four") and that different numbers represent different quantities. Preschool children also begin to recognize and name written numerals.

Counting

Counting is a fundamental skill in children's early understanding of numbers and quantities. It provides the basis for the development of number and **arithmetic** concepts and skills. Early on, children attempt to count everything around them, the number of steps on the way home, the cookies on their plate, or the number of blocks in a tower they built. This tendency to count everything is of considerable importance for the development of counting, as it provides the child with practice in learning the counting procedure.[5] At first children often omit some numbers when saying the list of number words or skip objects when counting. With repeated counting experiences and adult guidance at home and in preschool, children learn to apply counting skills precisely and use counting to determine the number of objects in a set.[6]

As teachers observe the children throughout the day, they are likely to encounter a great deal of spontaneous counting and reasoning about numbers: "One, two, three, five, seven, eight," the child may recite counting words when swinging outside. "Teacher, I am three," shares a child and counts, "One, two, three," showing three fingers. Preschool children's spontaneous counting and use of number words present teachers with wonderful opportunities to assess what children know and to facilitate their skills.

Sample Developmental Sequence
Counting

- ✔ Saying number words in sequence. May omit some numbers when reciting the number words. For example, the child's counting list may consist of the following number words: "one, two three, seven, eight, ten."
- ✔ Counts a small set of objects (five or six) but may have trouble keeping **one-to-one correspondence.** The child may point to more than one object when saying one number word or say a number word without pointing to an object.
- ✔ May count correctly a larger set of objects (about ten), keeping track of counted and uncounted objects by pointing and moving objects while counting.
- ✔ Understands that the number name of the last object counted (e.g., the number five when counting five objects) represents the total number of objects in the group (i.e., **cardinality**) and repeats this number when asked, "How many?"
- ✔ Knows to say the number words one-to-ten in the correct order, but is still learning the number sequence between ten and twenty. May omit some "-teen" words (e.g., 13, 14, 16, 18).
- ✔ Creates a set with a certain number of objects. For example, when asked to give three beads, the child counts out three beads from a larger pile of beads.
- ✔ Knows to say the number words up to twenty correctly.

Mathematical Reasoning in Action: Counting Ladybugs

VIGNETTE

Antonio was looking at a counting book, and in Spanish he counted the number of ladybugs in the picture, "Uno, dos, tres, cuatro." *Mr. Moises noticed him counting, repeated the Spanish counting words, and then responded in English, "Yes, four ladybugs: one, two, three, four." They moved on to the next page, and the teacher invited Antonio to count the ladybugs with him. The child counted in Spanish and the teacher then counted with him in English.*

TEACHABLE MOMENT

Counting books elicit spontaneous counting. Observing the children in the library area, the teacher is noticing that Antonio is counting in his home language. He encourages him to continue counting in Spanish and uses this opportunity to count with him in English. English learners need many opportunities to count in their home language and in English. For more information about strategies to support children who are English learners, see Chapter 5.

Mathematical Reasoning in Action: Who Has More Cars?

VIGNETTE

Playing with cars on the rug, a child argued, "I have more: one, two, three, seven, nine, ten." His friend replied, "No, I have more: one, two, three, four, five, six, seven." The teacher intervened and asked, "How do you think we can find out who has more cars?" "I count," said one of the children. The teacher suggested, "Let's count together," and she modeled counting together with the children. She put the cars in each set, in a row, and lined up the two sets against each other. The teacher pointed to each car while counting.

TEACHABLE MOMENT

Rather than telling children which one of them has more cars, she asks them for a solution (e.g., "How do you think we can find out who has more?") and lets them come up with a strategy to find out the answer (i.e., counting). She models for the children the use of counting. She also facilitates correct counting by putting the cars in each set in a row and by pointing to each car while counting. These strategies help children keep track of which cars were already counted and which cars are yet to be counted.

As illustrated in the above examples, preschool children's spontaneous counting and use of number presents learning opportunities, or "teachable moments." The teacher uses these spontaneous opportunities to facilitate and reinforce children's counting and mathematical reasoning. The teacher encourages individual attempts to count and reason about numbers and **scaffolds** as necessary, to introduce or reinforce mathematical concepts. The following strategies provide suggestions as to how teachers can develop children's understanding of number and quantity.

The following interactions and strategies promote preschool children's understanding of number and quantity:

Observe and listen to children's counts. Observe children's spontaneous counting and note their developmental level. Do the children tend to use a stable counting list? Can they recite the number words in the correct order? In what language? Up to what number? Can they keep track of the counted and uncounted objects while counting? Do they use counting as a means to quantify a set of objects? Are they comparing two quantities? Do they comprehend or use terms such as *more, less, same?* Observing preschool children's spontaneous counting and reasoning will enable teachers to assess and plan successfully and meet the needs of all children, including those with special needs. See "Sample Developmental Sequence of Counting" on page 242.

Encourage counting during everyday interactions and routines. Learning the sequence of number words in English involves the rote learning of the first 13 number words and later the rules for producing the subsequent "teens" number words and the beyond-twenty number words. This may proceed slowly and requires a lot of practice because learning number names in order takes practice. This does not mean that teachers should drill children to learn numbers. The teacher's modeling and the child's tendency to count and self-correct will facilitate the learning of the conventional sequence of number words. Encourage all children to count together as opportunities come up throughout the day. Children hear, say, and experience counting in the correct order over and over. Everyday interactions and routines offer numerous opportunities for counting and reasoning about number: during clean-up, "Everyone put five pieces away and then we'll be done"; in morning circle time, "How many children are wearing boots today?"; at snack time, "Please make sure every table has six apple slices"; during movement, "Let's jump seven times"; and at music time, counting while clapping with rhythm, "One, two, three, four, five."

Include preschool children's home language in counting activities, whenever possible. Use of the home language will reinforce counting skills and will show value for the child's home language and culture. Children who are English learners usually know how to count in their home language before they demonstrate the ability to count in English. In the beginning, they may not feel comfortable counting in English. Teachers should encourage them to participate and to count in their home language. Preschool English learners may need time to observe other children count in English before they feel comfortable taking an active part counting in English. For more information about strategies to support children who are English learners, see Chapter 5. For children who communicate in sign language, it is helpful to learn the number signs.

Ask questions that encourage purposeful counting. Use counting to determine quantity and answer a child's question within context: "I wonder how many stickers Ana has? One, two, three, four. She has four." To compare two quantities, the teacher might ask, "Which table has more children? How many more?" To create a set with a number of objects, the teacher could suggest, "Derek needs four sticks." Or to solve addition and subtraction problems, ask "How many blocks do we have altogether? How many are left?" Combine counting with pointing or touching objects to reinforce the concept.

Foster one-to-one correspondence within the context of daily routines. Preschool children practice **one-to-one correspondence** as they gather and distribute materials, such as placing one shovel in each bucket, giving one paper to every child, or as they help to set the table. Lunch helpers, for example, count out and distribute dishes, napkins, or fruit. The following dialogue between the teacher and the child helping to set the table before mealtime serves as an example.

Mathematical Reasoning in Action: More Cups

VIGNETTE

Mr. Raj asks, "Do we have one cup next to every plate?" Amy checks and says, "No, this one does not have one, this one does not have one, and this one and this one. We need more." Mr. Raj asks, "How many more do we need?" "Four . . . uh . . . no, maybe six. Let me count, one, two, three, four, five, six." Mr. Raj notices that she counted one of the plates twice and says to Amy, "Let's count again, slowly." He points to the plates that have no cups next to them and counts them one at a time with Amy, "One, two, three, four, five." Amy repeats, "Five." "Yes, we need five more cups," Mr. Raj answered. Mr. Raj helps Amy get five more cups and asks Amy, "Can you make sure we have one cup next to every plate?"

TEACHABLE MOMENT

Helping to set the table provided an opportunity to practice one-to-one correspondence (e.g., one plate, one cup) and to use purposeful counting (e.g., to find out how many more cups are needed). The teacher first let the child figure out the answer. When the child counted incorrectly, the teacher invited the child to count again and counted with her while pointing to each plate to facilitate correct counting.

Support preschool children's ability to apply the counting procedure. Counting the number of objects in a set means the child has to coordinate several distinct skills, reciting the number-word sequence while simultaneously keeping one-to-one correspondence between the objects being counted and the number words assigned to the objects. Preschool children may also tag or point to objects one at a time to keep track of those objects that have been counted and those to be counted. See "Sample Developmental Sequence of Counting" on page 242. Initially, preschool children are not skillful in applying the counting procedure precisely, but experiences with counting objects help them develop their counting skills. Those experiences may be spontaneous and informal and happen with teachers and with other children. Teachers can use the following strategies to gradually build preschool children's counting skills.

- **Provide lots of objects to count.** Provide preschool children with collections of small items to count such as, unit blocks, seashells, small figures, kernels of corn, or different sets of flannel pieces. Start with objects that are uniform in size, shape, and color so that children can focus on number without the distraction of other perceptual attributes. As children get more practice, they are ready to move to more abstract counting.
- **Start with small sets of objects.** Young children are more successful at counting small sets of objects. Provide children with small sets of objects (e.g., two or three), and gradually increase the number of objects that the children count.
- **Start with objects arranged linearly.** Young children are more successful applying one-to-one correspondence to linear sets of objects. When objects are arranged in a line, the beginning and end of the set are clearly marked, and children have an easier time keeping track of which objects were already counted and which objects are yet to be counted.
- **Model counting.** Point to, touch, or move each object aside as it is counted. Pointing to or touching each object as it is counted facilitates the one-to-one correspondence between the number words and the tagged objects during the counting process. Moving each object aside is also a helpful strategy for keeping track of which objects were already counted and which objects are yet to be counted.
- **Encourage children to self-correct their counts.** If children count incorrectly (e.g., skip a number or double count an object), invite them to count again: "Let's count again. More slowly, one . . ." and give them the opportunity to correct themselves.

Consider adaptations for children with special needs. Children with special needs may not move through the stages of counting as quickly as other children. Children with certain language impairments or hearing impairments have difficulty learning the sequence of number words and may show difficulty in developing counting skills.[7, 8] They would benefit from additional opportunities to count with adults and other children (e.g., with counting songs, finger plays, and games). Children with special needs would also benefit from combining words with actions to support counting. Marching or clapping while counting adds a kinesthetic dimension. Teachers could also support children with special needs by breaking the learning down into smaller steps, giving children small, manageable tasks (e.g., begin with counting a small number of objects with adults' help while counting). Children with physical disabilities may need to demonstrate mathematical knowledge in various ways. They do not necessarily need to engage in motor behavior and should be encouraged to use any means of expression and engagement available. Children with motor impairments may need assistance from an adult or peer to manipulate objects in order to count. Alternately, a child might demonstrate knowledge in these areas without directly manipulating objects. For example, a child might count verbally while a peer touches the objects. Children with visual impairments might be offered materials for counting that are easily distinguishable by touch. Their engagement is also facilitated by using containers or trays of materials that clearly define their workspace. (See Appendix D.)

Make number-related games, books, and other materials accessible to preschool children. Board games with a spinner, a die or dice, and other games such as dominos, number blocks, and cards and puzzles with numbers provide an engaging way to promote children's understanding of number and quantity. Children's books about numbers and counting can be used to introduce counting and basic addition and subtraction concepts. The teacher should include number-related books in the home languages of the children in the classroom. Books can be presented along with felt pieces or finger puppets to illustrate math content with action. Children benefit when teachers use props and gestures to act out, model, and demonstrate mathematical concepts.

Plan group activities focused on counting. Use large- and small-group activities to help children practice counting and use counting in meaningful contexts. Counting songs, finger plays, and children's books with numerical content provide a playful context for practicing counting and developing mathematical concepts. Preschool children enjoy counting as a group, especially when they are able to predict what number comes next as they count up or down the number list (e.g., *Ten Little Monkeys, This Old Man, Five Green and Speckled Frogs, Five Little Ducks, Un Elefante se Balanceaba, Chocolate, Los Numeros, Sé Contar del Uno al Diez).*

Mathematical Reasoning in Action: Singing and Counting

VIGNETTE

*There was **one** little bird in a little tree*
He was all alone, and he didn't want to be
So he flew far away, over the sea
and brought back a friend to live in the tree.

*Now there are **two** little birds, one little tree . . .*
(The song repeats as the number of birds increases by one)

A flannel board with felt pieces and small objects were used to act out the content of the song or story and to help children better understand the mathematical concepts. To illustrate the song above, the teacher first put one bird on the flannel board and then added a felt bird to the board each time the song repeated. The teacher paused and invited children to count. "How many birds do we have in the tree? Let's count together." The teacher also invited a child to lead the counting.

PLANNING LEARNING OPPORTUNITIES

Rhymes, finger plays, and songs with number-related content are common for practicing counting and introducing numerical concepts. Some children with special needs find it especially difficult to memorize the sequence of number words or understand the meaning of numbers. When number songs are introduced along with sets of items on the flannel board or magnet board, children have repeating opportunities to say and memorize the sequence of number words and to connect number words with quantity. Counting songs highlight the "number-after" relationship for the number word sequence and illustrate which of two adjacent numbers is a larger quantity (e.g., that four is more than three). Children enjoy counting together, especially when they are able to predict what number comes next as they count up or down the number list.

Beyond Counting: Recognizing and naming written numerals

The standard written numerals in our society are Arabic **numerals,** 1, 2, 3, and so on. Children see numerals all around the house, on the phone, the remote control, the clock, and in number puzzles, games, and counting books. They typically learn to recognize the symbols 1 through 9 sometime between the age of two and five with little difficulty.[9] The understanding of what numerals represent develops over time. Ongoing informal experiences with environmental print expose children to the link between number symbols (e.g., 1, 2, 3) and the different meanings. The numeral 5 next to five apples, for example, communicates the quantity of apples (a cardinal meaning). The numeral 5 in number labels on houses (e.g., the numeral 5 in the house address 15430), on a bus, and car license plate has a noncardinal meaning. To enhance the connection between numerals and the quantity they represent, the number symbols in the preschool environment should be accompanied by some representation for quantity whenever possible (e.g., 5 means that five children can be sitting around this table). Through everyday exposure to numbers and the use of numbers in meaningful situations, preschool children learn to identify number symbols (numerals) and to recognize the link between written numerals (1, 2, 3), numeral names (one, two, three) and numeral meanings.[10]

Integrate numerals into different areas of the classroom. Numbers are used everywhere around us. Teachers can incorporate numerals throughout the classroom in a variety of meaningful contexts: in the dramatic play area, the art area, the science area, the library area, and the sand and water area. Many items can be incorporated into different areas of the classroom, including items such as a calendar, a clock, a phone, a scale, a calculator, date stamps, address and phone books, rulers, measuring tapes, labels, and advertisements with numerals. In addition to exposing children to written numerals, such real-life items will familiarize children with everyday uses of numbers. Teachers may also include in the classroom different learning materials with numerals such as a number line, number blocks, magnetic numbers, and number stamps and cards. Consultation with specialists may be helpful to find materials that can easily be used by children with physical disabilities or other special needs.

Discuss numerals in print in a meaningful context. Refer to numerals in the environment as part of the daily activities: in books with numbers, the calendar, on labels or on measuring cups while cooking, or on the measuring tape when measuring height. Teachers can also encourage children to refer to number symbols in the environment through a search for numerals in the class (e.g., I Spy game). Ongoing experiences with environmental print will reinforce the link between the number symbols, their names, and their meanings.

Expose preschool children to quantities represented in different forms. Preschool children gain a better sense of numbers as they come across different representations of number in their environment. For example, "three," can be represented with three objects, three fingers, a pictograph, a number symbol (3), tally marks (III), or a pattern of dots (●●●). The teacher can expose

children to different representations of quantity through the use of finger play, tallying and graphing activities, games with dice or spinners, number cards, and dominos.

Promote use of the subitizing skill
Subitizing is the ability to quickly determine the number of items in a set, or in a pattern of dots, without counting. When children are presented with very small sets, they can tell "how many" without counting the objects in the set. Teachers should be aware of children's capacity to quantify small sets quickly. When children are asked "how many" with respect to small sets of objects, they may use subitizing and call out the answer and not necessarily count the objects one by one to find out how many. Teachers can provide children with opportunities to apply subitizing in everyday situations by asking children for the number of objects in a small set; referring to the number of objects in small sets (e.g., "There are *three* chairs in that corner," We have *two* orange slices in this bowl"); conversing with children; or when pointing to pictures in children's books.

2.0 Understanding Number Relationships and Operations

Preschool children develop the conceptual basis for understanding number relationships and operations. When comparing two small sets, they can recognize whether the sets are equal or not. They can also recognize the ordering relation of two sets that are unequal and identify which set has more or less. Experiences with number-change transformations such as "adding to" and "taking away" provide the conceptual foundation for solving simple arithmetic problems. Young children understand that addition increases the number of items in a set, and subtraction decreases the number of items in a set. They use counting strategies to solve simple addition and subtraction problems.

Young children also develop a basic understanding of **part–whole relationships,** as they recognize that parts can be combined to make a whole, and a whole quantity can be broken down into two or more parts. Experiences with part–whole relationships and the decomposition of numbers into smaller groups (e.g., decomposing "six" into "four" and "two") support the understanding of number relationships and operations and from the conceptual basis for future understanding and solving of missing-addend problems (__ + 3 = 5), and multidigit addition and subtraction problems (e.g., 11 + 2 = __). Preschool children can be informally introduced to number relationships and operations through everyday interactions, language and literacy activities, and games.

Research Highlight

Research indicates that the ability to reason about numbers starts as early as infancy.[11] Five-month-olds show sensitivity to the effects of addition or subtraction of items on a small collection of objects. Toddlers viewing three balls put into a container and then one being removed know to search for a smaller number of balls, and many search for exactly two balls.[12]

By the time children are in preschool, prior to having any formal lesson in arithmetic, they use a variety of strategies to solve simple addition and subtraction problems.[13] They may use manipulatives or fingers to represent the numbers in the problem and count out loud to find out the answer. As they get older, they rely less and less on finger counting. To solve an addition problem such as 4 + 2 presented with concrete objects (e.g., color crayons), the child may count all objects "one, two, three, four" and then continue with the second set of objects "five, six" and find out there are a total of six. At a later stage, the child may **"count on"** from the second set of objects. Knowing the number of objects in the first set (e.g., "four"), the child starts with "four" and continues to count "five, six" to find out the total number of objects, rather than starting to count from "one" with the second set of objects.

Mathematical Reasoning in Action: Playing with Balls

VIGNETTE

While outdoors, a small group of children were playing with balls, throwing them into a net basket that was on the ground. Mr. Phan was standing by, watching the children taking turns throwing balls into the basket. When all six balls were inside the basket, he took them out one at a time. "One, two, three, four, five, six," he counted while handing the balls back to the children, and they again threw the balls into the basket. Julia, one of the children in the group, was standing by the basket watching the balls go in. When all balls were in the basket, she helped Mr. Phan take out the balls and hand them back to the children. Julia started to keep track of the number of balls that went into the basket. "One," she shouted after the first ball went in, "two" after the second ball was in, and so on. When five balls were in the basket, Julia said to Mr. Phan, "One more, and then we take all the balls out again." Mr. Phan asked Julia, "How many balls would we have altogether after the last ball goes in?" Julia answered, "five, six . . . six balls!"

TEACHABLE MOMENT

Outdoor play provides numerous opportunities for counting and reasoning with numbers. Children often count while swinging back and forth, while passing a ball from one child to another, or when climbing up the steps to go on the slide. Teachers may use these opportunities to count with children and think about numbers while playing. Children, particularly English learners, need a variety of meaningful and exciting counting and arithmetic experiences, in which they can combine counting with actions and model a problem with concrete objects. In this example, Mr. Phan made children aware of the total number of balls they were shooting to the basket. It created a goal for children: all six balls in the basket before the children continued to the next round of throwing. Julia, in particular, enjoyed watching the balls adding up in the basket. She became aware of the number of balls already in the basket and those yet to be thrown by children. Julia and other children in the group have learned through play about counting, quantity (the meaning of six), part–whole relationships, and addition.

Mathematical Reasoning in Action: How Many Boys? How Many Girls?

VIGNETTE

The teacher comments to Jennifer, "Let's find out how many children are here today. Jennifer, would you please help me count the girls?" Counting the girls, Jennifer says, "One, two, three, four, five, six." Then she announces six. The teacher responds, "We have six girls. Now let's count the boys. Brian, would you please help me find out how many boys are here today?" Brian counts, "One, two, three, four, five . . . five," and holds up five fingers. The teacher says, "We have six girls and five boys. Do we have more boys or girls?" Most children call out, "Girls." One child said, "Boys." Another child replies, "No, it's girls because six is more than five." The teacher holds up six fingers and counts, "One, two, three, four, five, and one more is six. We have five boys and six girls. We have more girls than boys. Can you help me find out how many children we have altogether?" The teacher counts together with the children, pointing to every child while they are counted: "One, two, three, four, five, six, seven, eight, nine, ten, eleven. How many?" The children call out, "Eleven." The teacher says, "We usually have twelve children, but today we have only eleven. Can you help me figure out how many children are not here today?"

PLANNING LEARNING OPPORTUNITIES

A daily routine activity, such as checking attendance during morning circle time, serves as the context for introducing and practicing several mathematical concepts and skills, illustrating part–whole relationships (e.g., six girls and five boys, eleven altogether), comparing quantities by counting, doing arithmetic, and practicing counting skills. The teacher asks children probing questions and encourages them to find the answers. The teacher combined the verbal or gestured responses (e.g., five, six, eleven) with the visual representation (e.g., fingers) to assist those children who need more than one modality for learning.

The following interactions and strategies promote preschool children's understanding of number relationships and operations:

Promote the use of comparison terms *(more, same as, fewer,* or *less)* through everyday interactions. Everyday situations provide many opportunities to explore number relationships. Encourage preschool children to use comparison terms such as *more, fewer,* or *same as,* when comparing numbers in the everyday environment. "We have more boys than girls," "Both of you have the same number of stickers." "This table has fewer oranges." The word *fewer* is used to describe a smaller number. The word *less* should be used to describe a smaller amount or degree: "This bottle has less water," "There is less play dough on this table," "There are fewer crayons in this box." Use comparison terms in everyday situations to help children learn the meaning of such words.

Use everyday interactions and routines to illustrate and discuss addition and subtraction transformations. At a very young age children understand that "adding to" results in more, and "taking away" results in less. Building on children's natural understanding of these concepts, teachers can introduce children to simple addition and subtraction problems through everyday routines. "You have three cars. Can you give Andrea one? How many cars do you have now?" "You have three stickers. If I give you two more, how many stickers would you have altogether?" See the "Research Highlight" on page 251.

Mathematical Reasoning in Action: More Crackers

VIGNETTE

During snack time, Veronica asked: "Can I have two more crackers?" The teacher replied, "Yes, and I see you already have two crackers. When I give you two more, how many crackers will you have altogether?"

TEACHABLE MOMENT

In this situation, during snack time the child asks for more crackers and the teacher recognizes the opportunity to reinforce the mathematical concept of addition. The teacher presents the child with an "adding-to" arithmetic problem (2 + 2 = __) and uses concrete objects (e.g., crackers) to solve the problem.

Introduce preschool children to the concepts of addition and subtraction through literature, songs, and games. Stories, songs, and games provide a playful way to introduce "adding-to" and "taking-away" operations, and part–whole relationships. Experiences with concrete sets of objects, in particular, can illustrate for children addition and subtraction concepts and enable children to solve simple addition and subtraction problems by counting objects. For example, when telling the flannel board story *Rooster's Off to See the World,* the teacher says, "One rooster met two cats," and she places one flannel rooster next to two cats and asks the children, "How many animals do we have altogether?" The story continues, "The rooster and the cats met three frogs." The teacher places three flannel frogs, "One, two, three," next to the rooster and cats and asks, "How many do we have now?" The flannel board story provides the context for introducing the "adding-to" concept. For ideas on adapting books for children with physical disabilities, please refer to the Literacy section on pages 106 and 107.

Make estimations. Encourage preschool children to estimate: "How many balls do you think are in this jar?" "How many seeds are inside the apple?" "How many steps are outside the door?" When possible, ask children to count and check their estimate. Children enjoy this process very much. Invite children to estimate in a group setting and record their estimates. It illustrates to them that different children have different estimates. Making an estimate and then counting "to find out" is a powerful and effective way to facilitate children's understanding of number and quantity.

Use graphing with children. Encourage preschool children to collect data, tally totals, and graph the results. Children enjoy taking an active part in this process. Invite children to collect and record numerical information (e.g., the number of children who have pets, the number of people in each child's family). Create a chart or graph, using real objects, to represent numerical information collected by children. Discuss with children the information presented in the graph. Graphs lead naturally to making comparisons: "Which group has more?" Which group has fewer?" "Can you tell without counting?" "How many more are in this group?"

Bringing It All Together

Bagel Shop

While singing the "Bagel Shop" song, children count to solve an arithmetic problem. One of the children plays the role of the Baker. The rest of the children take turns buying bagels at the bagel shop and help the Baker find out, "How many bagels are left in the bagel shop?" It begins with the teacher placing bagels on an upright flannel board, and the children count to determine how many bagels are on the board. Next, the class sings the following number song, and the teacher invites one of the children to buy bagels.

"Five *little bagels in the bagel shop*
Sprinkled poppy seeds on the very top
Along came (child's name) with two
pennies to pay
He bought two bagels and
walked away."

The child buying bagels gives the Baker two pennies, and the Baker takes away two bagels from the board and gives them to the child. The remaining bagels are visible, and children are asked to predict how many bagels are left in the bagel shop. Each time the song repeats, children first predict the answer to a problem (e.g., five "take away" two), and then check their prediction by counting. The value obtained from checking their prediction then serves as the start for the next round (e.g., "Three little bagels in the bagel shop . . ."), and the singing and selling procedure repeats. At the beginning of the year, the teacher may start with a small number of bagels (e.g., five or six) and subtract only one bagel at a time. Over time, the teacher may alter the problems' difficulty level by gradually increasing the number of bagels (e.g., from seven to fifteen) and varying the number of bagels being removed (plus or minus one, two, or three).

For example, the teacher of older preschoolers placed seven bagels on the board. The children called out, "I think there are seven"; "No, eight"; "Seven, I counted"; "I counted, too; it's six." The teacher invited one of the children to count the bagels. She arranged the bagels in a row and pointed to them one at a time, as the child was counting to help keep one-to-one correspondence between the number words and the bagels being tagged. The child counted, "One, two, three, four, five, six, seven," and the class agreed that there were seven bagels. Next the class sang the song, and one of the children bought two bagels. The children were asked "How many bagels are left in the bagel shop?" Most children said five, but some said six. The teacher said, "Some of you think we have six bagels, and many of you think there are five bagels. How can we find out?" She invited one of the children "to count and find out." (Based on an example described in a research journal.[14])

During the Bagel Shop activity, children are counting and doing arithmetic in the context of a real-life setting. The subtraction problems are presented with concrete objects (e.g., taking away bagels), and counting serves a purpose. Children count to check the prediction and solve an arithmetic problem. It enhances the meaning of counting and facilitates children's problem-solving and arithmetic skills. A group-learning experience in which children take turns counting and reasoning is also an opportunity for the teachers to observe and learn about individual children's understanding of number. A context that represents a real-life setting, in particular, makes mathematics more engaging and fun, as children experience different roles in buying and selling bagels. Think of other real-life settings you can bring into the classroom or arrange outside to provide the children with a meaningful context for counting and doing arithmetic (e.g., a grocery store, shoe store, a train with a conductor collecting tickets from passengers).

Engaging Families

The following ideas may help families to develop their children's number sense:

- ✔ **Communicate to parents the broader meaning of number sense.** Teachers may need to explain to parents how they can support children's development of number sense. Often what parents know about mathematics education is based on their own school experiences and how they were taught.[15] Their view of mathematics at this age is often restricted to children being able to count to high numbers and to recite basic arithmetic facts. Some may ask for pencil-and-paper activities with numbers for their children long before children are ready. Teachers need to communicate to parents the broader aspects of developing number sense; for example, using counting in real-life situations, comparing numbers and discussing which is more or less, making estimations (e.g., How many grapes are in this bowl?), and solving simple addition and subtraction problems. The teacher should explain to parents that such meaningful experiences lay the foundation for a basic understanding of mathematical concepts for later learning of more advanced ones. She might share with parents how children are engaged in counting and reasoning with numbers in the preschool environment. Parents may try to apply similar ways to engage children with numbers at home.
- ✔ **Remind parents that daily use of numbers can become learning experiences for children.** Numbers are everywhere: in the house, on the way to school, in the grocery store, and in sport games and outdoor activities. Parents can point children to numbers and talk with them about what numbers are used for as they go about their everyday experiences. They can encourage children to count and to solve problems related to number. For example, children can count coins for purchases at the store, count the number of plates and cups when helping to set the table, count the number of crackers in the bowl, and divide them equally to two groups in order to share with a friend. Parents can talk with children about mathematical ideas. "You have five pennies, and we need seven. How many more pennies do we need?" "How do you know you both have the same number of crackers?" "How many seeds do you think are inside this apple? Now I cut it open. Let's find out." Parents should ask questions of their children rather than just telling them the answer.

✔ **Provide number-related games and books.** The teacher can also encourage parents to choose books from the local library that involve numbers and to play with children number-related games such as cards, dominos, puzzles, or board games. Parents can also use common games to engage children in counting, addition, and subtraction. For example, while playing mini-bowling at home, children can count and find out how many pins they knocked down and how many are still standing, with each turn.

Questions for Reflection

1. What have you included, or could you include, in your environment to support the development of children's counting and understanding of number?
2. Think about your group's everyday activities and routines. In what ways can you develop children's counting skills in the context of everyday routines?
3. How do you engage children in comparing numbers and use terms such as more, fewer, or same as?
4. Think about the children in your group. How do you learn about the counting and reasoning skills of individual children in your group? How do you support individual children in developing number sense? How would you modify the Bagel Shop activity to make it work for children with varying abilities?
5. What real-life settings can you set up in your preschool environment to provide a context for counting and doing arithmetic?

Algebra and Functions
(Classification and Patterning)

One may wonder how **algebra** is related to young children, as it may bring up the thought of traditional high school algebra. Obviously, preschool is not the time to teach traditional algebra, but this is the period when foundational algebraic concepts evolve and gradually develop. Children observe the environment and learn to recognize similarities and differences. They learn to sort, group, and classify objects. They learn to recognize ordering relations, such as large to small, and to identify patterns. They develop the ability to make predictions, form generalizations, and derive rules. Experiences with classification and patterning during the preschool years allows for the development and practice of algebraic thinking and reasoning—skills essential in learning mathematics and science.

Teachers have a key role in promoting preschool children's classification and patterning skills. They can:

- draw children's attention to patterns in the environment;
- set up patterning and classification experiences;
- discuss with children their sorts and patterns; and
- encourage children to come up with their own patterns or ways of classifying objects.

Teachers' interactions with children, as they classify or work with patterns, not only facilitate the children's math skills and introduce them to math vocabulary, but also provide a vehicle for language development. "How are these the same?" "Here you have all the red triangles and here all the yellow ones." "Look at the colors. Can you see a pattern?" "This is a big pile of round leaves." "It seems like you separated the rocks into two groups." "How are they different?" (e.g., smooth and bumpy). Interactions of this kind provide children with the descriptive words they need to describe their ideas and attach meaning to their actions. The interaction is especially relevant for children who are English learners because such interactions allow them to infer the meaning of words used by the teacher or peers as they classify objects or describe a pattern and expand their vocabulary in English. Children are introduced to math concepts as well as to new vocabulary in meaningful and engaging contexts as they **sort, classify,** and make **patterns.** The next chapter describes some strategies teachers can apply to promote the classification and patterning skills of all children, including those with disabilities or special needs.

1.0 Classification

Young children naturally engage in classification activities as they separate and group things with similar **attributes** (e.g., same color, size, shape) or belong to the same **class** or category (e.g., dogs, chairs, airplanes). The process of forming a class based on similar attributes starts at infancy, as children continual form classes based on their ability to recognize "sameness" of members in a group. For example, a child may first refer to any swimming animal as "fish," but over time they see sea animals that are not fish. The notion of "fish" in the child's mind is revised, and new classes such as "whales" and "dolphins" are created. Classification involves giving descriptive labels to the feature(s) used in sorting. It facilitates the child's acquisition of concepts and language and allows children to explore their environment and organize information in an efficient way.[16]

Mathematical Reasoning in Action: Collecting Leaves on a Nature Walk

VIGNETTE

As part of a curriculum unit on the seasons, the children went for a nature walk and collected various types of leaves. During the walk and later in the classroom, the children explored the leaves and were encouraged to describe different attributes of the leaves such as shape (pointy, round, long, needle), size (small, tiny, wide, big), color (red, green, yellow, orange, brown) and texture (smooth, soft, hard, wet, dry, rough). Children were then asked by the teacher to sort the leaves: "Put leaves that belong together in groups."

The teacher asks Enrique, "Why did you put these leaves together and those leaves together?" Enrique responds, "They are same." The teacher asks, "How are these the same?" Enrique points and says in Spanish, "Café aquí, amarillo aquí, y hojas rojas." ("Brown here, yellow, here, and red leaves here."). The teacher points to each group of leaves and says in English, "Great! Brown, yellow, and red leaves. What other ways can we sort the leaves? How about putting all the big leaves here and all the small ones there?" The teacher models for the child, sorting leaves by size. "Where do you think this leaf would go?"

PLANNING LEARNING OPPORTUNITIES

As part of children's process of exploring different attributes of the leaves, the teacher invited children to engage in a **sorting** activity. Enrique, a child who is learning English, is encouraged to explain his sorting, as are other children in the group. The teacher used this experience to introduce him to new vocabulary in a context that was meaningful for him and that allowed him to determine the meaning of the English words. For more information about strategies to support children who are English learners, see Chapter 5.

The following interactions and strategies support children's classification skills:

Organize the classroom into different categorized storage areas to facilitate classification. The basic organization of the preschool environment illustrates for children how different objects in the world belong together. Blocks of different sizes and shapes are stacked on the block shelves, books are organized in the library area, and dolls of different kinds are in the dramatic play area. Items of the same category can be sorted into subcategories. The wood blocks can be arranged by shape and size. Toys such as cars, airplanes, and trucks can be stored in separate boxes. Animal figures (e.g., farm animals, ocean animals, wild animals) or art supplies (e.g., crayons, pencils, markers) can be kept in separate containers. The music instruments can be organized by instrument families: percussion, string, and brass instruments. Organizing and labeling the classroom environment so that objects and materials that belong together are stored together facilitates children's classification skills through everyday interactions.

Include materials and objects for sorting in the environment. In addition to using the natural environment as an opportunity for many sorting and classifying activities, teachers may want to provide certain materials to sort and classify or place in a pattern: rocks, shells, seeds, leaves, buttons, beads, wheels, plastic counters of different shapes, fruits, and cubes. Teachers may want to rotate items related to the current topic of interest in the classroom. For classification, it is important to provide items that belong to the same group yet vary by one or more identifiable attributes (e.g., color, shape, size, function, texture, or visual patterns). Tools may be provided to help sort and classify, such as trays, containers, egg cartons, or cups.

Identify opportunities for sorting and classifying in everyday routines. Sorting and classifying is a natural part of the daily routine in preschool. Clean-up and other times, such as when recycling materials, setting the tables for snack or lunch, and choosing activities to explore, are natural opportunities for children to sort and classify. An organized classroom environment turns clean-up time into a sorting experience. Putting away the blocks, toys, materials, tools, and instruments requires children to think about the attributes of the items or their function and to store items together that "belong together" based on different criteria. Pictorial labels enable all children to sort toys and restore materials to their proper place. Teachers may give children support and encouragement and model for children where things go, "The rectangle blocks all go together on this shelf," the teacher gestured toward the shelf while talking. "All the crayons go together in one box, and all the pencils in another box," the teacher pointed to the box while talking. "Here is the basket for the farm animals, and here is the basket for the wild animals," says the teacher as he puts one animal in each basket. Modeling with action is particularly helpful for children who are English learners or children with hearing impairments.

An environment that is organized with areas neatly labeled facilitates sorting and makes clean-up time a learning experience. In addition to clean-up time, everyday routines provide other natural opportunities to sort and classify. Setting the table for snack and lunch may present valuable sorting experiences. As preschool children help set the table, they may set out only the small bowls, not the big ones; separate the forks from the spoons; or sort the snack foods. For example, the teacher may put slices of green apples in one bowl and slices of red apples in another bowl. Recycling is yet another unique opportunity, as children sort their trash into labeled bins for plastic, paper, glass, and aluminum cans.

Recognize sorting in play. During child-initiated play when children choose their own activities, they may sort and group objects to help organize their play activities. They may sort out the triangle blocks and the long rectangle blocks when building a castle, sort the red and yellow beads to create a necklace, sort the firefighters or other figures in pretend play, and so on.

Engage preschool children in conversations about their sorting and classifying. Teachers have a key role in making the sorting and grouping experiences meaningful and rich with language for *all* children. Interactions with children will help them express verbally or by some other way, such as sign language, their criteria for sorting and will provide them an opportunity to explain their reasoning. Observe children, and note where children are developmentally and what vocabulary they are using.

– **Ask questions.** Ask children to explain and describe their sorting and classifying. "It seems that you have two groups of animals. Why did you put these animals together and those animals together?" "Tell me how you sorted these rocks." "What name could you give this group?"
– **Help children label the groups and verbalize their criteria for sorting.** Use simple sentences or phrases that provide the children with the descriptive words they need to expand their descriptive language: "You have the whales on this side and the dolphins on the other side." "This is a big pile of triangles." "Which ones are the wild animals?" "Here are the red cars, and there are the cars that are not red."
– **Encourage children to come up with their own criteria for sorting.** As children engage in sorting and classifying, teachers can encourage them to come up with their own criteria for sorting: "Can you sort these into groups that belong together?" "Can you sort these another way? How would you do it?"

Plan opportunities for preschool children to sort and classify. In addition to spontaneous opportunities to sort and classify objects in the environment and during ongoing routines, teachers may want to plan specific sorting and classifying activities.

- **Plan for children at different levels.** Simple, basic sorting activities are a good start. Choose appealing objects as well as those that are easy for young children to grasp and manipulate. A variety of small objects are offered on a tray, and children sort the objects into their corresponding containers labeled with a picture or a sample object. Begin with objects (e.g., blocks or crayons) that vary by only one attribute (e.g., crayons differing only in color). They make it easier to sort and classify. Continue with objects such as buttons, beads, or dried beans of various colors, sizes, and shapes. Providing sorting activities at different levels allows teachers to meet the needs of all children, including children with disabilities and special needs. Giving children manageable tasks builds their confidence and allows them to experience successes. If the group includes young children or children with cognitive delays, think carefully about the size of the objects as some children may still be exploring items by putting them in their mouths.

- **Integrate sorting into children's current topic of interest and study.** Any collection of objects can be sorted by some criteria. Sorting activities, therefore, can be an integral part of children's exploration and study of any topic: pumpkins, apples, leaves, animals, tools, vegetables, and so on. Use classroom materials and experiences that reflect children's natural environment and culture and relate to children's interests. As described on page 260 in "Mathematics Reasoning in Action: Collecting Leaves on a Nature Walk," when children sort objects that they currently study and explore, sorting becomes more interesting and meaningful.

2.0 Patterning

Patterning, like classification, involves the child's natural tendency to organize information in the environment. It requires the child to observe discrete elements, recognize similarities and differences, and make generalizations. From a young age, children see patterns around them: on toys, clothing or quilts, and in nature. Their daily routine creates a pattern, and they listen to songs that follow patterns (e.g., "The wheels on the bus go . . . The people on the bus go . . ."). By preschool age, although children may recognize patterns in the environment, they may not always draw them with symbols or create their own patterns. For instance, they may clap or jump in a pattern (e.g., clap-clap-hop-hop), or use beads to create a red-blue-red-blue pattern. Young children appreciate the predictability that comes with patterns. They enjoy being able to predict what comes next, and they notice immediately if someone breaks a pattern (e.g., "But we always have snack first and then go outside").

Research Highlight

Compared with classification skills, relatively little is known about the development of young children's patterning skills. The Berkeley Math Readiness Project[19] examined the informal patterning knowledge of low-income and middle-income children and the effect of curricular intervention on their patterning skills. The study revealed some key findings that can help teachers in planning ways to effectively support the patterning skills of all children.

- **Identifying the core unit of a pattern is a challenge for all children.** The majority of prekindergarten children attending preschool, regardless of socioeconomic background, experienced difficulty with identifying the core unit of a pattern at the beginning of the preschool year.
- **Pattern extension is a later development than pattern duplication.** Both middle-income and low-income groups were significantly better on pattern duplication (e.g., using blocks to make a pattern that "looks just like this") than on pattern extension (e.g., presented with two repetitions of the pattern, children were asked to finish making the pattern).
- **Positive effect of curricular activity on patterning knowledge.** Both middle-income and low-income groups exhibited significant progress in their ability to duplicate a pattern correctly after participating in a patterning curriculum activity. Low-income children had more difficulty duplicating a simple pattern correctly than did middle-income children.

A **pattern** is a regularly repeated arrangement of things such as numbers, objects, events, or shapes. Young children may begin with a simple pattern (e.g., red-yellow, red-yellow, or circle-circle-square-square), keeping the number of elements in the repeating unit constant (e.g., *one* red item, *one* yellow item). More complex patterns may vary the number of each element (e.g., red-yellow-yellow, red-yellow-yellow; *one* red element, *two* yellow elements) or include more than two items in the repeating unit (e.g., a pattern with three elements: red-yellow-blue, red-yellow-blue).[17] Preschool children readily identify and duplicate patterns in their environment, but extending or creating patterns may require more guidance from adults.[18] The following strategies provide suggestions as to how teachers can help children develop their abilities to identify, describe, replicate, extend, and create patterns using various modalities throughout the day.

Mathematical Reasoning in Action: Making Bracelets

VIGNETTE

A small group of children were making bracelets by stringing different color beads together. The teacher commented, "You can choose any design you want for your bracelet. Ana created a pattern that looks like yellow, purple, yellow, purple." A few minutes later the teacher pointed to a bracelet that was created earlier but left on the table and said, "Look at this pattern: green, green, red; green, green, red. What do you think comes next?" The children became more engaged with patterns. One of the girls was trying to replicate her friend's pattern; others wanted to create their own patterns.

PLANNING LEARNING OPPORTUNITIES

The teacher planned an activity in which she deliberately introduced the concept of patterns. The small-group setup allows her to support children individually. She encouraged children to extend a pattern (e.g., "What comes next?") and supported them in the process of replicating and creating patterns.

To facilitate patterning skills of all children, including those with special needs, the teacher might at first limit the colors available to keep patterns simple and then gradually increase the range of colors. In this example, the teacher demonstrated for the children the concept of pattern and described the pattern created.

Mathematical Reasoning in Action: A Hunt for Patterns in the Outdoors

VIGNETTE

During outdoor play, the teacher invited a group of children to join her in a search for patterns. The teacher approached children with excitement: "Remember how we looked at a picture of a caterpillar in the book, and we saw a pattern? black, yellow, white—black, yellow, white. If we look really carefully around us, we may discover many different patterns. We may find patterns in leaves, in the trunks of trees, in flowers, even in creatures such as bees or butterflies. Look at this leaf, for example. What do you see happening over and over?" Maya looked at it closely and said, "It has the lines, the lines on it again and again." "These lines are called veins," the teacher explained and showed the veins in the leaf to everyone in the group. The veins create a pattern. Can you find veins in other leaves around us? Let's go hunt for more patterns and see what we find."

PLANNING LEARNING OPPORTUNITIES

The teacher used the outdoor learning environment to discover patterns with children. Children learned that patterns are around them and can be found in the natural environment. The teacher has mentioned a previous experience they have had with patterns to grab children's attention and remind them what patterns are about.

The following interactions and strategies support the development of patterning skills:

Point out patterns in the environment. Daily routines bring natural opportunities to discover patterns with children. Circle time, transition time, mealtime, and free time all become opportunities to spontaneously discover and talk about patterns. Patterns are part of the physical environment: in toys, books, on the carpet, walls or fences, on clothing and accessories. "Look at the carpet. Can you see a pattern?" Children may explore the bark on a tree, the veins in leaves, the colors in a rainbow, or the patterns in bees, zebras, butterflies, caterpillars, or snakes. Specific items in the classroom environment can present children with patterns in meaningful contexts. A calendar, for example, can illustrate for children the pattern of the days in a week or months in a year. The class schedule can illustrate for children how certain things repeat every week on the same days (e.g., every Monday Brenda's grandmother comes to preschool to read books with the children, and every Wednesday Ms. Santos comes to play the piano). Older preschool children enjoy playing a pattern hunt to see who can identify the largest number of patterns in the classroom.

Engage preschool children in conversations about patterns. Conversations with children will help them identify and analyze the discrete elements in a pattern and will facilitate their ability to recreate and extend patterns.

- **Say the patterns aloud as a group to build the rhythm of repetition:** "Red, green, yellow, red, green, yellow, red, green . . ."
- **Ask questions.** Promote children's thinking about patterns: "What would come next?" "What happens over and over again?" "Do you see a pattern?" "Is this a pattern? Why?"
- **Help children describe patterns and use descriptive words:** "First green, then red and then yellow, again and again." "Two squares, one triangle, two squares, one triangle . . ." "Tell me about the order of these colors."

Mathematical Reasoning in Action: Building a Fence with a Pattern

VIGNETTE

While playing with blocks, Joseph was sorting out the long and short rectangles: "I am making a fence."

The teacher noticed and said, "Long rectangle, short rectangle, long rectangle, short rectangle," touching the blocks while talking. "Joseph, look at your fence. You have a pattern. What is happening over and over again?" After Joseph completed the fence, the teacher suggested, "Do you want me to get you paper and a pencil so you can draw your pattern to save in the pattern book?"

TEACHABLE MOMENT

Observing Joseph playing with blocks, the teacher has noticed a pattern created by Joseph and brought it to his attention. The teacher described the pattern aloud, asked a leading question (e.g., "What is happening over and over?"), and used the term *pattern.* The teacher also offered Joseph the opportunity to draw his pattern. The teacher has created a "pattern book" with photos and drawn pictures of found and created patterns. Children look up patterns, find previous patterns, and compare them to new patterns.

Plan for children at different levels. Start by having children identify and duplicate patterns. Extending or creating patterns may be more difficult for children. You may choose to begin with simple patterns. For example, introduce children to patterns with two elements in the repeating unit (e.g., red-blue, red-blue or apple-apple-pear-pear, apple-apple-pear-pear). Have the same number of each element in the repeating unit (e.g., *one* red item, *one* blue item or *two* apples and *two* pears). At a more advanced level, you may introduce patterns with more than two items in the repeating unit (e.g., a pattern with three elements: apple-pear-orange, apple-pear-orange), and to patterns with varying number of each element (e.g., red-blue-blue, red-blue-blue; *one* red followed by *two* blues). See the Research Highlight on page 264.

Play with patterns in various formats. Patterns can be presented in different formats: through movement, sound, language, objects, or pictures. Expressing patterns through different modalities (kinesthetic, tactile, auditory, and visual) provides different learning modes and facilitates preschool children's understanding of patterns. Variety also helps ensure that pattern experiences are accessible to all children, including those with disabilities or special needs. There are numerous opportunities for children to *duplicate, extend,* and *create* patterns through art, music, movement, literacy, and science.

– **Patterns with objects and pictorial designs.** One common way to create patterns is with concrete objects such as blocks, counters, beads, interlocking cubes, shapes, or other small objects. Objects that can be identified by touch provide tactile input for children as they duplicate, extend, and create patterns. For example, children can use plastic beads of different colors and shapes to make a bracelet with repeating patterns, or they use a variety of toppings to decorate a celery stick with a pattern. Different artistic expressions (e.g., sponge painting) may lend themselves to the expression of patterns such as when a design is repeated over and over. Children can create a desired pattern of flowers and then plant flowers in the garden duplicating this pattern. Children can also duplicate patterns they observe in the natural environment. For example, they can observe a caterpillar and record its pattern of colors.

– **Patterns through movement.** Children can experience patterns in a physical way. Teachers may invite preschool children to create patterns physically through marching, standing, sitting, jumping, or clapping (jump-jump-clap-clap, jump-jump-clap-clap or stand-clap-sit, stand-clap-sit). Often these are duplicated while singing a song (e.g., "If You're Happy and You Know It, Clap Your Hands," or "Hokey Pokey"), or through games. For example, in playing Simon Says, children are invited to duplicate and create different patterns (e.g., Simon says, clap, clap, stomp"; "Simon says, clap, clap, touch your knee").

– **Patterns with sounds.** Preschool children can create patterns with different sounds by using rhythm instruments such as shakers or sticks. For example, they can vary the volume of sound to create a pattern (loud-loud-soft, loud-loud-soft) or create a pattern with different sounds (bell ring-shake-shake, bell ring-shake-shake).

– **Patterns through rhymes and stories.** Many nursery rhyme songs and stories have repetitive structures, phrases, or rhymes that form patterns (e.g., "The Wheels on the Bus," "Old MacDonald Had a Farm"). Many children's stories include repeating patterns. Children can easily grasp the repetitive structure and carry it to the next verses in the story. They especially have fun predicting what comes next once the pattern is identified. For example, the text in the book *Brown Bear, Brown Bear What Do You See?,* by B. Martin Jr. and Eric Carle, has a predictable pattern.

Bringing It All Together

Sorting, Counting, Graphing and Comparing Apples

During circle time, the teacher shared with the children different varieties of apples: Red Delicious, Granny Smith, Golden Delicious, and Fuji. Children discussed the features of the varieties of apples. The teacher asked the children in the classroom to bring their favorite apple from home for the next day. All the apples brought from home were put in a basket, and the children were given time to play with the apples in one of the activity areas. The basket of apples immediately sparked mathematical reasoning and problem solving by the children. The children sorted the apples first by color.

The teacher observed the children and posed questions or made comments along the way. "So here you put the green apples and here the red apples, but what about these apples? Can you think about another way to sort the apples?" The children sorted apples by variety, by color, and even by taste (sweet versus sour). Children were curious and enthusiastic. "How many apples are in the basket? How many green apples? How many red?" The teacher asked, "What is the class's favorite kind of apple?" How do you think we could find out?" After the children sorted the apples by variety, the teacher asked, "Which group has the most apples?" and the children counted the apples in each group.

During circle time, the teacher discussed the findings with the children. Together with the children, the teacher graphed the data, one column for each group of apples, using unit blocks. Then the children were encouraged to compare the columns and discuss, "Which is the class's favorite and least favorite kind of apple?" "Do more children like red apples or green apples?"

The teacher capitalized on children's natural mathematical skills and interest in the topic. Bringing their favorite apple from home made children even more engaged and enthusiastic. The basket of apples sparked children's interest, and the teacher developed it progressively into a mathematically rich experience. The teacher gave children time to explore the apples, commented on children's categories, and posed thought-provoking questions. The experience with apples illustrates the integration of math, science, and language learning and how several mathematical skills and concepts such as sorting, counting, comparing, collecting data, and graphing can come together.

(*Note:* If the program policy does not allow bringing food from home, or if some families are unable to provide an apple for their child to bring to school, a basket of apples could be provided by the school for this activity.)

Engaging Families

The following ideas may help families to develop children's classification skills:

- ✔ **Explain to parents about classification and patterning.** Teachers may need to explain to parents what classification and patterning are about and how they contribute to children's understanding of mathematics. Parents who are informed of these developing skills are more likely to engage children in classification and patterning experiences in their everyday routines.

- ✔ **Create classification experiences outside the preschool environment.** The teacher can give parents some ideas about classification and patterning activities that children can experience outside the classroom. Just as the preschool environment illustrates for children how different objects in the world belong together, so does the organization of items at home, at the grocery store, or other places (e.g., vegetables and fruits are sorted and presented by kind; all cereal boxes are placed on the same shelf and so forth). Children learn how different things belong together by observing their environment.

  Children have many opportunities at home to sort through objects and to look for similarities and differences among them. At a very young age, they may play with sorting toys, for example, sorting circles, squares and other shapes and inserting them in the matching opening, square in the square opening and so on.

  As they grow older, parents can engage children in different sorting activities around the house. Children can sort clothes by color or type (e.g., colored shirts together, white socks and towels), sort shoes or socks and find pairs that belong together, or unload groceries and sort into different piles (e.g., boxes on the counter and cans on the table).

- ✔ **Create patterning experiences outside the preschool environment.** Children also enjoy identifying things that repeat in their environment. Parents may draw children's attention to patterns in designs and pictures, in furniture, wallpapers or rugs in the house, or in songs and in children's books. For example, they may ask children to find and describe patterns in their clothing ("My shirt has a pattern. It is yellow, blue, yellow, blue"), or in picture books or magazines. They may also sing songs with children or read books with repeated rhyming phrases, emphasize the repeating phrase, and let children predict what comes next (e.g., "One, Two, Buckle My Shoe," "Head, shoulders, knees, and toes"). Such books and songs reinforce patterns through words, sound, and movement, and are playful ways for children to practice language and mathematics skills.

Questions for Reflection

1. How do you, or could you, organize your classroom environment to facilitate classification skills?
2. How could you integrate sorting and patterning experiences into children's current topic of study?
3. What sorting or patterning activities would you, or do you, offer children who are just beginning to grasp these concepts?
4. How do you engage children in exploring and describing patterns?
5. How do you, or could you, use classification and patterning experiences to develop children's language and introduce them to new vocabulary?

Measurement

Young children develop an intuitive notion of measurement through natural everyday experiences. They explore and discover properties such as length, height, volume, and weight as they look for a longer block, measure who is taller, pour sand from a small bucket to a larger one, or try to pick up a heavy box and ask for help. They make comparisons to see which is longer, taller, heavier, larger, or smaller. Teachers should build on preschool children's emerging concepts of measurement and provide experiences that facilitate the development and learning of these concepts. This practice does not suggest teaching young children how to measure in inches or pounds using measuring tools. It is about exploring and describing the height, weight, or size of objects, comparing and ordering objects by different attributes, using comparison vocabulary, and measuring with standard and nonstandard units. Exploring and reflecting on comparison and measurement sets preschool children on a path for developing a formal understanding of measurement later in school.

1.0 Compare, Order, and Measure Objects

The Measurement strand encompasses three main measurement concepts:

- **Comparing:** Children develop an understanding of attributes (weight, size, volume) by looking, touching, and directly comparing objects. They can determine which of two objects is longer by placing two objects side by side, or which of two objects is heavier by picking them up. Learning the vocabulary to describe objects (e.g., heavy, big, short) and to compare them by different attributes (e.g., "This is heavier," "Mine is bigger") is fundamental for acquiring the concepts of measurement. Older preschool children can start to compare objects, indirectly, using a third object (e.g., use a paper strip to represent the length of one object and then lay the strip against the other object).
- **Ordering:** As children explore and compare objects, they can also identify ordering relationships. For example, they can arrange three or more objects by size from smallest to largest. This requires children to observe and distinguish slight variations of the attribute and order the objects in a progressive sequence (e.g., small bear, medium bear, large bear).
- **Measuring:** Older preschool children begin measuring the length of objects, often by using nonstandard units (e.g., a block). For example, to measure length with a nonstandard unit such as a block, children position many same-size blocks along the object they measure, from end to end, without leaving a space between the blocks, and count the number of blocks. They may find out, for example, that the table is seven blocks long. Such experiences develop children's understanding of the nature of units.

Research Highlight

A research-based instructional approach suggests that teachers of young children follow a developmental sequence in helping children develop concepts and skills of length measurement.[20] First, informal activities establish the attribute of length. Children should be given a variety of experiences directly comparing the size of objects to determine equality or inequality of length and develop concepts such as "longer" or "shorter." Only then are young children ready to learn to measure and connect number to length. Research emphasizes the importance of solving real measurement problems in which children explore principles of measurement such as identifying a unit for measure and placing that unit end to end alongside the object without leaving space between successive units (referred to as *unit iteration*).

The traditional approach holds the view that children measure length with rulers only after a long experience with nonstandard units and manipulation of standard units. Recent research suggests that children as young as six years old are capable of and benefit from using rulers.[21] Children typically learn to measure with rulers in early primary grades. However, exposure to ruler and measuring tape is appropriate throughout the preschool years.

Mathematical Reasoning in Action: Playing at the Water Table

VIGNETTE

Playing at the water table outdoors, Sara was filling up different-size containers with water. The teacher, Ms. Frances, noticed that Sara had filled up a cup with water and poured the water into a bigger container over and over. Ms. Frances commented, "It looks like you are using the cup to fill up this pitcher with water. How many more cups do you think it would take to fill it up?" Sara looked at the pitcher and said, "I don't know, maybe three, no . . . five." Ms. Frances suggested, "Let's find out together how many cups of water it will take to fill up this pitcher all the way to the top." Together they filled up the cup and poured the water into the pitcher, one cup at a time, while counting, "One, two, three, and four." Sara said, pleased with herself, "It took four more cups. I said five; I was almost right."

TEACHABLE MOMENT

By pouring water or sand from one container into the other, children learn about the **volume** or **capacity** of different containers. While observing Sara and other children at the water table, the teacher has recognized the opportunity to make it a learning experience of estimation and measurement. The teacher illustrated for the child that measurement involves the total number of the repeated equal-size unit (e.g., one cup).

Children benefit from repeated measurement activities and the use of measurement vocabulary. The following interactions and strategies provide suggestions as to how teachers can help preschool children, including children, with disabilities or special needs, build measurement concepts and skills.

Provide opportunities to promote measurement concepts in the environment. The indoor and outdoor learning environments have plenty of things that can be measured. Children even compare themselves to each other and to other objects: "This plant is taller than me." Objects in different sizes, such as buckets, shovels, balls, blocks, brushes, or cups, present more tangible opportunities to compare and order objects by size. Unit blocks, for example, are especially good for measurement explorations. An environment rich with measuring opportunities should also include standard measurement tools in different interest areas, although measurement using standard units is not the primary focus in preschool. By providing an environment with measuring tools, a teacher encourages children to become familiar with them and to explore their function. Rulers and tape measures may be part of the block area, although they can be used for different purposes throughout the day. Height charts can be used to measure and track growth over time. Measuring cups, spoons, and scoops can be used in the sandbox and for any cooking activity, real or pretend. Tools to measure weight, such as a balance scale, a produce scale, or a bathroom scale, can be part of the dramatic play area. A thermometer can tell children the temperature outside or inside. Preschool children are not expected to know how to read and use these tools, but with teacher guidance they learn that specialized tools measure different attributes. The teacher can gradually increase the number of measurement tools in the environment. Not all tools should be introduced at once, and children should be given time to explore tools, learn about their function, and apply that knowledge in their play. In addition, materials for measurements in nonstandard ways may be included, such as ribbon, yarn, paper clips, same-size block units, yardsticks, or unit blocks. See the “Research Highlight” on page 273.

Observe preschool children’s measurement concepts in everyday play and routines. Children placed two trains next to each other to see which is longer. The teacher commented, “How do you know which train is longer?” Another child filled up a bucket of sand. The bucket became too heavy, and the child could not pick it up. She poured out some of the sand and tried again. Preschool children learn about measurement concepts through everyday play. Such interactions with objects teach children about length, capacity, weight, and other measurable attributes. They constantly make comparisons: “My train is longer,” “I have a bigger shoe,” “Let’s see who is taller.” While observing children at play, teachers can listen to the words they use in English and in their home language to describe and compare objects, learn about their level of measurement concepts, and find out more about their interests. This valuable information will help teachers design learning opportunities that are developmentally appropriate, meaningful, engaging, and accessible to all children in the group.

Mathematical Reasoning in Action: Which Is Taller?

VIGNETTE

In the block area, a group of children built a block tower. Cathy says to the teacher, “Look how big it is. It is reaching me up to here.”

The teacher says, “You built a tall tower. Which do you think is taller, the table or the tower?”

Cathy replies, “The tower.”

The teacher asks, “Why do you think the tower is taller?”

Cathy responds, “Because look, the table only reaches me up to here,” pointing to her waist, “and the tower is up to here,” pointing to her chest.

TEACHABLE MOMENT

The teacher is asking questions to direct Cathy’s attention to the height of the objects. Cathy is interested in finding out “which is taller” and came up with a way of answering this question. She is using herself as a reference point to determine which is taller: the block tower or the table.

Facilitate and reinforce measurement concepts in everyday play and routines. Everyday interactions provide teachers with opportunities to help preschool children identify and compare attributes of length, area, weight, and volume.

- **Build preschool children's descriptive and comparison vocabulary.** Describing and comparing attributes not only attunes children to the idea of measurement, but it also expands children's vocabulary in a natural and functional way. Point to the objects as you describe and compare them. Children who are learning English, in particular, need to hear measurement-related vocabulary used in context in order to comprehend the meaning. Model the use of comparison vocabulary when talking with children. "This is a very *tall* tree. Which tree do you think is *taller?*" "Your lunch box is very *heavy* today. It is *heavier* than mine." "Look at the *long* train you built. Let's see whose train is *longer.*" "This is a *big* box. I think we a need a *bigger* box." For more information about strategies to support children who are English learners, see Chapter 5.
- **Ask questions.** Teachers' questions direct children's attention to measurable properties of objects, facilitate the child's thinking about measurement concepts, and model the use of measurement vocabulary. "Which ribbon is longer?" "Which beanstalk is taller?" "Which is heavier, the foam block or the wood block?" "Which container holds more?" Questions should be short and simple for children who are learning English and children who are just beginning to learn those concepts. The teacher should ask meaningful questions that reflect a desire to really understand the child's thinking and not overwhelm him with lots of questions. Use questions sparingly and purposely and allow children time to think and respond.
- **Challenge preschool children to use measurement to solve problems.** As teachers observe children at play, teachers pose questions to enrich children's experiences. "How far can you jump?" "How can we find out how big this rocket ship is?" Children enjoy exploring and finding the answers to such questions. Teachers can help children figure out ways to measure, using standard or nonstandard units. For instance, the teacher may suggest that the children measure the distance of their jump with same-size blocks or model for children how to use a tape measure to measure the length of a slide or the circumference of a watermelon. See the "Research Highlight" on page 273.

In addition to preschool children's natural measurement experiences through everyday play, teachers can provide rich and meaningful measurement experiences in small or large groups to target different measurement concepts.

Provide opportunities to compare and order objects. Comparison lays the foundation for understanding measurement. Offer children opportunities to compare objects based on size, weight, or capacity. For example, children can explore a pair of pumpkins and determine which one is larger. A balance scale may be used to compare the weight of different objects. Talk with children about why the scale is moving up or down. "Which object is lighter?" "Which object is heavier?" After comparing two items, children can compare three or more items and put them in order. For instance, with three pumpkins of different sizes, children can place them in order from largest to smallest and from smallest to largest. Teachers can engage children in ordering, using a variety of objects in the preschool environment, such as sticks, buckets, balls, nesting cups, or blocks. Children can arrange sticks from shortest to longest or the buckets in the sandbox from "holds the most" to "holds the least."

Use literature to illustrate measurement concepts. Read storybooks such as *Goldilocks and the Three Bears* to children that emphasize measurement concepts. Teachers can have children act out the stories and use different size objects to create the setup (e.g., large, medium, and small bowls, beds, and chairs). This allows teachers to illustrate the concept of size and use comparison vocabulary in a meaningful context: "Which bowl is largest?" "Can you put these three bears in order from smallest to largest?"

Provide small-group activities using standard and nonstandard measurement. Plan activities that show the child the need for measurement and use concrete objects to illustrate the measurement process. Preschool children enjoy finding answers to measurement questions, especially if the activity expands what they already know and is related to what is familiar to them. For some activities, teachers can illustrate the use of nonstandard measurements for children such as multiple copies of objects of the same size (e.g., wood blocks, unit blocks). For other activities, teachers can model for children the use of standard tools in real-life situations. Children who are English learners may already know a concept but need the English words or measurement-related vocabulary to describe it. Describe in words, using gestures and concrete objects, the measurement question and the process of measuring. For more information about strategies to support children who are English learners, see Chapter 5.

Some common activities that involve measurement include planting and cooking. In planting, for example, children can use same-size sticks to keep equal distances between plants (e.g., "How far apart should the plants be?"). They can use a three-inch string to find out if each hole is deep enough and track and compare the growth of plants over time. Children may keep a daily log of the plant's growth. While cooking, teachers may invite children to help measure by using measuring cups and spoons. "Now we need one cup of flour. Can somebody help me measure?" "I need half a teaspoon of cinnamon." The children can help the teacher identify the appropriate measuring tool, and the teacher can use it to demonstrate for the children how to measure exact amounts.

Children can also use measurement in exploring the body. For example, the teacher can create with children a cutout of their foot. Children estimate the length of their foot, and then the teacher shows them how they can measure the exact length using a measuring tape. Children may be able to record what they have measured. The teacher can also invite children to compare their height by creating a bar graph. He may discuss with children, "How tall are you?" "Who is the tallest?" "Who is the shortest?" If the class includes a child who does not yet walk, teachers can encourage children to measure height while the person lies down. This technique works well for tall teachers too.

Encourage preschool children to estimate measurements. A measurement experience could start with estimation before doing the actual measuring, "How many scoops of sand do you think we need to fill this green cup?" "About how many blocks will cover the distance from here to the table?" Encourage children who are English learners to express themselves in their home language. Estimation focuses children on the attribute being measured, helps develop familiarity with standard units, and motivates children to measure and find out how close they came to their estimates.

Encourage preschool children to record and document what they have measured. Recording their measurements allows children to convey information about the process and the outcome of their measurement, using drawings, numbers, and words. Teachers can transcribe for children their observations and explanations. Children with special needs and other disabilities may use alternate methods of communication, if needed. For example, some children may use sign language, pictures, or a computer. Keeping records of their measurements allows children to compare their measurements to others or to their own measurements over a period of time (e.g., tracking growth).

Bringing It All Together

Tracking the Growth of Sunflowers

As part of exploring and learning the concept of growth, the children have planted sunflower seeds in the garden. A long stick was attached to each plant, and the teacher asked that every week the children mark on the stick the height of the sunflower. Tracking the growth of sunflowers has generated comparison and measurement experiences. For example, one week the teacher pointed to one of the sunflowers and explained to the children, "Last week when we measured this sunflower, it was up to here. It was seven inches long. This week it is up to here. How many more inches do you think it grew in the past week? What is your estimate?"

Children were encouraged to make estimates and then were invited to measure the growth of this sunflower. "How can we measure how much it has grown since last time?" Children had different ideas. Some children said, "You need a ruler." Others said, "With this" and pointed to a measuring tape. Over time, children were also comparing the sunflowers one to another. On one occasion, the teacher helped a small group of children compare the height of two flowers by using a string to represent the height of one flower and then laying the string against the second flower.

Children enjoyed tracking the sunflowers' growth and finding out, "Which sunflower is taller?" and "Which is taller?"—the child or the sunflower.

Tracking the growth of sunflowers generated opportunities for children to compare, estimate, and measure length. Measurement was a natural part of this experience, and it illustrated for the children its application in everyday life. The teacher facilitated and reinforced measurement concepts by asking questions, encouraging children to make estimates, challenging children to use measurement to answer questions, and supporting children's efforts in measurement.

Engaging Families

The following ideas may help families to develop children's measurement experiences:

- ✔ **Communicate to parents the importance of talking with children about measurement.** Explain to parents how early measuring experiences set the foundation for developing a formal understanding of measurement later in school. Early measuring experiences with parents or other family members will expose children to measurement terms and comparison vocabulary in the child's home language. Parents can invite children to compare the length, height, area, or weight of different objects. For example, they can play with children a game in which children have to find objects around home that are "longer than," "heavier than," or "taller than" a particular object. Teachers can invite parents and other community members to the classroom to take part in cooking, gardening, building, or other activities involving measuring.

✔ **Encourage parents to involve children in everyday measurement experiences.** Measurement is a practical math skill used in many different aspects of everyday life. Cooking, sewing, gardening, grocery shopping, a visit to the doctor or to the post office—all involve measurement. Encourage parents to include children in everyday measuring experiences and to talk to them about what they are measuring. Parents may need to be reminded of the many opportunities they have throughout the day to talk with children about measurement and to demonstrate for children the use of different measurement tools; for example, a scale to measure weight, a measuring tape or a ruler to measure length, and a thermometer to measure temperature. Explain to parents that young children may become familiar with these tools but are not expected to know how to read and use these tools without adults' guidance. Also communicate to parents that preschool children can measure length by using nonstandard units. Some examples of measurement experiences children had in class using nonstandard units coould be shared so that parents join children in measuring the length of different objects at home, using the child's own unit of measurement (e.g., "The table is six lunch boxes long").

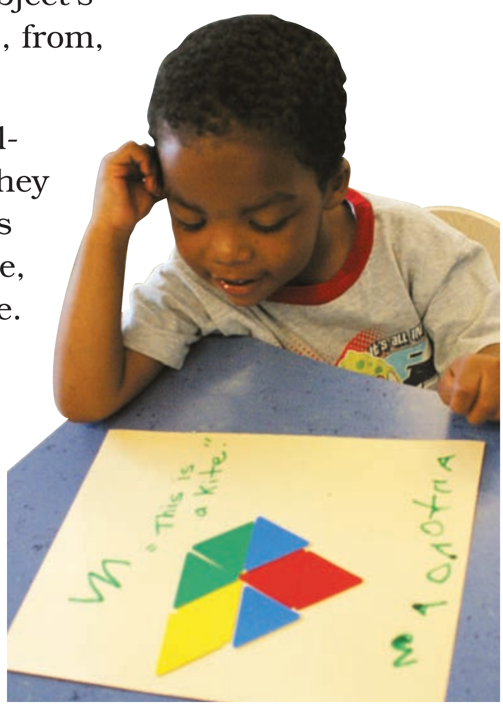
Questions for Reflection

1. What other preschool experiences can you think of that invite children to practice measurement naturally?
2. What could be added to the physical environment to promote children's learning of measurement concepts and skills?
3. Think about the group's daily routine. In what situations could you model the use of comparison vocabulary (e.g., heavier, smaller, longer, shorter) when talking with children? How would you support English learners in learning and using comparison vocabulary?
4. How could you integrate measuring experiences into your group's current topic of study? What measuring experiences would be ones that are relevant to the culture, interests, or life experiences of the children in your group?
5. How could parents be part of measurement activities in your classroom (e.g., in cooking, building, woodworking, gardening)?

Geometry

Geometry is the study of shapes and spatial relationships. Children enter preschool with a strong intuitive knowledge about shapes, spatial location, and transformations. They learn about geometry as they move in space and interact with objects in their environment. From infancy they begin to form shape concepts as they explore their environment, observe shapes, and play with different objects. Before they can name and define shapes, very young children are able to match and classify objects based on shape. During the preschool years, children develop a growing understanding of shape and spatial relationships. They learn the names of shapes and start to recognize the attributes of two- and three-dimensional shapes. They also develop an understanding of objects in relation to space, learning to describe an object's location (e.g., on top, under), direction (e.g., from, up, down) and distance (e.g., near, far).[22, 23]

Teachers have a vital role in expanding children's thinking about shapes and space. They should ask questions and provide materials that encourage children to explore, describe, and compare shapes and positions in space. The use of questions and access to materials is particularly helpful for children who are English learners to gain an understanding of geometry concepts while developing second-language skills. The following strategies will guide teachers in actively supporting preschool children's development of geometry concepts and spatial sense.



1.0 Shapes

Learning about shapes goes beyond merely knowing the names of common shapes. It involves the exploration, investigation, and discussion of shapes and structures in the classroom.[24] Experiences with two- and three- dimensional shapes allow children to learn to notice individual attributes and characteristics of shapes and to identify similarities and differences among them. Rich experiences with shape lay the foundation for more formal geometry in later years. Teachers should help children develop a deeper understanding of shapes by encouraging them to explore shapes and their attributes and providing opportunities for children to represent, build, perceive, and compare shapes.[25]

Mathematical Reasoning in Action: Discovering Shapes with Blocks

VIGNETTE

Mr. Gerry notices Amelia building wall of blocks and moves closer to observe. Amelia says to Mr. Gerry, "I need one."

Mr. Gerry responds, "Do you need another square block?"

Amelia responds, "Yes, but no more."

Mr. Gerry says, "I have an idea for you. We can use this triangle block and this triangle block to make a square." He hands Amelia two triangle blocks. "Put them facing each other just like this," he says, demonstrating for Amelia.

Amelia tries to put together the two triangles to make a square and says, "I don't know how."

Mr. Gerry responds, "Yes, you turn around this triangle and put it by the other triangle. Now look at these two triangles facing each other. What does it look like?" "A square!"

Amelia says, surprised and giggling, "I need it here."

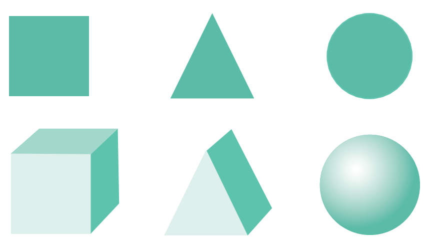
Mr. Gerry says, "Now you finished building the wall."

TEACHABLE MOMENT

Amelia has recently joined this preschool group. She is learning English. Mr. Gerry noticed that Amelia was looking for another square block, and the teacher used the opportunity to interact with the child. The teacher has introduced Amelia to the names of shapes (e.g., square, triangle) and has illustrated for her how a geometric shape can be composed from other shapes.

The following interactions and strategies promote understanding of shape concepts:

Refer to shapes and encourage the use of shape names in everyday interactions. Preschool children learn the correct names of shapes by hearing others call objects by their geometric names. Children first identify simple two-dimensional shapes such as circle and square. Over time, they learn to identify and describe a greater variety of shapes (triangle, rectangle, hexagon, trapezoid). Children play with three-dimensional shapes (e.g., building blocks) and, while in preschool, they may also begin naming and describing some three-dimensional shapes (e.g., sphere, cube).



The teacher should use the correct shape names during everyday interactions and routines. Describe what you see and point to or touch objects when saying their shape names. Oral descriptions are particularly important for preschool children who are English learners. Help children understand mathematical terms by using extensive modeling, accompany words with gestures (e.g., pointing or tracing shapes in the air), point to objects, act out terms, and use short, clear sentences. For more information about strategies to support children who are English learners, see Chapter 5.

- During play, the teacher observed the children's construction with blocks and commented, "I see you used the rectangle blocks to make a wall." "Look, you put a small triangle on top of a small square," the teacher pointed to the objects while saying the shape names. "You built a little house."
- At breakfast, the teacher asked, "What shape is the pancake?" Later, at snack time, the teacher announced, "Today, we are having crackers that are circles and crackers that are rectangles."
- When reading a storybook, the teacher commented, "Look at this snowman. What shapes can you see?"
- When playing with blocks, the teacher asked, "What shape is the block you have on top of the long rectangle block?"

Engage preschool children in conversations about shapes. Preschool children recognize geometric shapes by their overall physical appearance, but they do not yet think about the attributes or properties of shapes. For example, children may recognize a square because it looks like a square, but not think of it as a figure that has four equal sides and four right angles. Teachers can draw children's attention to the attributes of different shapes by discussing with them the parts and attributes of shapes and by encouraging them to build and represent shapes in many different ways.

- **Encourage preschool children to observe and compare shapes:** "Can you find another rectangle around the room?" "How are they similar?" "Here is a square and here is a triangle. How are they different?"
- **Talk about shapes and discuss their attributes:** "Let's find out how many straight sides are in a rectangle." The teacher counts while pointing to a rectangle, "One, two, three, four. How many straight sides do we have in a triangle? Can you help me find out?" Oral descriptions are particularly

helpful for children with vision disabilities as well as those with visual-spatial challenges.

A child's world is filled with shapes in different sizes and positions, but observing shapes in the environment is not enough to build a full understanding of shape. Preschool children need to explore, manipulate and represent two- and three-dimensional shapes in a variety of ways. Hands-on experiences offer the best learning opportunities for all children, including those with special needs. The following experiences will expose children to the attributes and properties of different shapes.

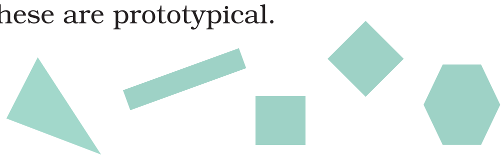
Provide materials that encourage preschool children to explore and manipulate shapes in space. It is crucial that the classroom environment include materials that encourage children to manipulate and represent shapes in a variety of ways. Children with disabilities and other special needs, in particular, need lots of hands-on sensory experiences with three-dimensional shapes and real-life objects in a variety of shapes and textures. The preschool environment should have a variety of blocks of different shapes, colors, sizes, and thickness. In addition to blocks, shapes may be provided in a variety of forms in all areas of the classroom: interlocking plastic shapes, shape containers for sand and water play, sand molds, shape sponges, cookie cutters, stickers, magnets, shape templates, geoboards, and beads in different shapes.

Include books, games, and other learning materials with shape-related themes in the preschool environment. Share with preschool children books about shapes (e.g., *The Shape of Things, The Village of Round and Square Houses).* As teachers read aloud, point to pictures and discuss with children the names and attributes of shapes. Books can also show children objects from different perspectives and give meaning to spatially related words (e.g., *under, in, on).* Another playful way to introduce shapes is through shape-related games such as shape lotto, shape bingo, and puzzles. Some preschools may choose to incorporate a software-based mathematical curriculum. There is computer software that allows children to perform action on shapes such as flipping, sliding, and turning shapes in different angles. Software of this kind engages children in a variety of shape-related activities (e.g., solving a shape puzzle, creating a composition of shapes) on the computer.

Provide preschool children with playful opportunities to explore and represent shapes in a variety of ways. To develop their concepts of shape, young children need to handle, explore, manipulate, and create shapes in a variety of ways. Hands-on experiences offer the best learning opportunities for all children, including those with special needs. The following experiences will attune children to the attributes and properties of different shapes.

- **Play with blocks.** Preschool children learn about shapes by touching, moving, putting together, and taking apart. Block play provides endless opportunities to learn about shapes. When playing with blocks, children perceive three-dimensional shapes from different angles and discover relationships between shapes. Children can see how a circle fits between arches, two right triangles form a rectangle, and two squares form a rectangle. Children with motor impairments may need assistance from an adult or peer to manipulate objects in order to explore two- and three- dimensional shapes. A child might also use adaptive materials (i.e., manipulatives that are easy to grasp). Children with visual impairments need materials that are easily distinguishable by touch. Their engagement is also facilitated by using containers or trays of materials that clearly define their workspace.
- **Match, sort, and classify shapes.** Provide children with a collection of shapes varying in size and color. Ask children to put all the same kind of shapes together. Discuss with children why a shape belongs to a group.
- **Create and represent shapes.** Help preschool children represent shapes in a variety of ways. For example, children can create two-dimensional shapes from small items such as beads, sticks, or strings or with play dough, clay, flexible straws, or pipe cleaners. They can also form shapes with their bodies. You may invite children to pair with a friend and use their legs or fingers to create circles, triangles, squares, and diamonds. Children also enjoy drawing, tracing, and copying shapes. Preschool children draw shapes, especially when drawing a picture of a house, a person, or a tree. Teachers may offer children solid cutouts to trace shapes or encourage them to trace a shape in the air using their fingers. Tracing helps children to focus on critical attributes of each shape.
- **Compose and decompose shapes from other shapes.** Provide children with different shapes (e.g., squares, triangles, trapezoids, and rhombus) and let them use these shapes to form other shapes. For example, they can use two same-size right triangles to form a rectangle or to form another triangle.

Present preschool children with many different examples of a type of shape. Children have mental images of shapes—visual prototypes created by the culture—through books, toys, games, and other materials. Triangles are usually equilateral (i.e., having all the sides equal) and isosceles (i.e., having at least two sides equal) and have horizontal bases. Rectangles are usually horizontal and elongated. These are prototypical.



Children should be exposed to many different examples of a shape. Examples of triangles and rectangles should include a variety of shapes, including long, thin, and wide, varying in orientation and size.[26] Draw children's attention to atypical shapes and encourage them to describe why some nonstandard examples belong to a category (e.g., "This is also a triangle, but it is thinner"; "This is a long rectangle, and this is a short rectangle").

2.0 Positions in Space

When young children crawl through a tunnel, climb up a ladder, go under a table, move forward in a wheelchair, or swing up and down, they develop a sense of spatial relationships. As they move their bodies in space, they learn position, direction, and distance relationships between their bodies and other objects and between different objects in space. Preschool children need to learn many words to be able to describe and name positions and directions in space (e.g., *in* *and out, top* and *bottom, over* and *under, up* and *down, forward* and *backward,* and *around* and *through).* Teachers can help children develop spatial vocabulary, especially children who are English learners, by using and demonstrating the meaning of spatial words during daily activities.

Mathematical Reasoning in Action: Moving Through an Obstacle Course

VIGNETTE

It was a rainy day, and the children could not go outside. The teacher had set up an indoor obstacle course for the children. She demonstrated how to use the course, talking as she moved. "It starts here. I crawl under this table. Next, I jump over this pillow. Then I crawl through this tunnel. Next, I hop across the rug, and finally I walk in between the chairs."

PLANNING LEARNING OPPORTUNITIES

Children learn spatial orientation through physical activity. Setting up an obstacle course provided the teacher with the opportunity to introduce and demonstrate the meaning of spatial orientation skills and vocabulary. As children went through the course on their own, the teacher described their actions, using spatial orientation terms (e.g., "Stay low as you crawl under the table; remember, you have to hop across the rug").

The following interactions and strategies can help children develop skills related to spatial relationships:

Provide materials and equipment to promote spatial sense. Young children develop their spatial sense through movement and interactions with objects in the environment. Outdoor equipment designed for large-muscle activity fosters children's spatial sense. Climbing up the monkey bars, going up and down a slide or on a swing, and driving a bike or a scooter around the play yard all help children develop their sense of position, direction, and distance. Preschool children also explore space by building a three-dimensional complex construction or a maze by using large boxes, blocks, geometric shapes, cardboard, even chairs and tables. They enjoy getting in, out, over, or under their construction. In movement, teachers use hoops, beanbags, or balls to introduce positions in space in playful ways (e.g., throw the ball up, jump in and out of the hoop, or put the beanbag under the arm). Children with motor disabilities and visual impairments need supported opportunities to experience spatial relationships in order to develop their spatial sense. Specialists working with individual children have specific ideas of how to support and promote the learning of this important concept.

Support preschool children's spatial sense in everyday interactions. As children experience concepts such as far, on, under, and over, they should learn vocabulary words to describe these spatial relationships. English learners may know the concepts and the corresponding vocabulary in their home language, but they need scaffolding to learn spatial vocabulary in English. Teachers may want to use simple concepts and vocabulary first (e.g., *in, on, under, up, down),* and then introduce more complex concepts and vocabulary (e.g., *in front, behind, beside, between).* For more information about strategies to support children who are English learners, see Chapter 5. Children with speech and language disabilities may need many opportunities to practice this vocabulary as they join with their peers in play.

– **Use spatial words and point out spatial relationships.** Point out spatial relationships naturally during play. Teachers may give directions, ask questions, or simply make comments (e.g., "Can you please put all the markers in the box?" "I see you put the beanbag on your head." "Let's see who can jump over the pond").
– **Expand preschool children's words.** Encourage children to use spatial words. When children try to describe position, direction, or distance, teachers expand on their ideas and demonstrate for them the use of spatial words in context. For example, if the child refers to his construction and says, "Look, I put the big block here like this and it doesn't fall." You may demonstrate for him the use of spatial words, "Oh, I see you put the big rectangle block on top of many small blocks."

Provide preschool children with planned experiences to promote the understanding of spatial sense. Plan small- or large-group activities to enhance children's understanding of spatial concepts and introduce spatial vocabulary words.

– **Songs and games.** Sing songs and play games that direct children to move their bodies in space. For example, "Simon says, put your hands on top of your knees, jump up and down, hold the beanbag behind your back . . ."
– **Literature.** Read aloud stories that use position words (e.g., *above, below, up, down).* Point to pictures and illustrate for children spatial positions with actions. After the reading of a book, children can act out the story and use position words to describe the characters' actions.
– **Construction.** Provide children with opportunities to organize materials in space in three dimensions using construction toys (e.g., interlocking cubes) or scrap materials. Teachers can also build with the children an obstacle course or an outdoor maze. Children experience themselves in space by going through, over, around, and in and out of different things.

Bringing It All Together

Building a Castle

The teacher had noticed that several children in her group had shown a strong interest in castles. They built castles in the block area, in the sandbox, and even looked for castles in fairy tale books when visiting the library. The teacher suggested that the group build a big castle outside. They started by gathering the materials. The children brought from home different-size boxes and figures or characters to be included in the castle. The teacher also offered big cylinders, cones, building blocks, construction boards, and other materials. The children made different suggestions: "Put all the big boxes here and the small ones on top of them." "I put it above this for the roof." "We can use these for the tower."

The teacher described their ideas using names of shape and spatial terms. "So you want to put the small square blocks on top of the big rectangle blocks." "Are you suggesting using the cylinders to build the tower?" The children enjoyed building the structure, using different shapes and materials, and were proud of it.

During circle time, the teacher invited children to describe the castle and how it was built. "Look at the castle you built. Can you tell me what it looks like?" Children were encouraged to use spatial words and the names of shapes in their talk. The activity evolved into a long-term project. The children kept adding more pieces to the structure and added different elements to decorate the castle.

The teacher presented a topic of interest to the children in the group. The castle project exemplifies how children can learn about geometry concepts by physically touching, moving, and putting together objects of different shapes. In the process of building the castle, children were encouraged to use the names of shapes and the words to describe spatial relationships (e.g., *above, below*). The teacher has made it a rich learning experience by offering children objects in a variety of shapes, observing children in their work, describing children's ideas in words, asking questions, and inviting children to observe and describe the castle in their own words. The project not only facilitated increasing the children's knowledge of shape and spatial concepts, it also promoted collaboration work and creativity.

Engaging Families

The following ideas may help families to develop children's awareness of geometric shapes:

✔ **Encourage parents to refer to shapes in the environment when talking with children.** Parents and other family members can support children's development of geometry concepts through everyday interactions with children. Teachers should encourage parents to refer to shapes in the environment when talking with children, "Look at your pancake. It's a circle. We can use this rectangle pan to bake this cake." When parents and other family members talk with children about shapes, they illustrate the concept of shape and introduce children to the names of different shapes in their home language. Parents can also help children learn names of shapes by playing games. For example, play I Spy and have children look around the house and identify as many items of a certain shape. When driving, or on the bus, parents can use traffic signs as an opportunity to identify and describe shapes. “Look at this yellow sign. What shape is it?” “The stop sign is red. It is the shape of an octagon. It has eight sides. Let’s see if you can find another stop sign.” In addition to identifying and naming shapes, children should explore and describe shapes. The teacher should communicate to parents that children learn best about geometry concepts through hands-on experiences. Holding and manipulating objects of different shapes, building with blocks, drawing and tracing shapes, creating shapes with play dough, or doing a puzzle all help children learn about the characteristics of different shapes.

✔ **Encourage parents to use spatial words in everyday interactions with children.** Parents use spatial words to describe position and direction in space in everyday interactions and play with children (“I am right *behind* you,” The book is *on* the chair,” “Put the shoes *under* the bed”). Parents should be aware of children’s opportunities to experience and describe themselves in space using words such as *above, under, up, down, in* and *out.* By listening to parents and other family members using these words, children will have a better understanding of spatial concepts and will learn spatial vocabulary in their home language. Children will start identifying themselves in space by using spatial words (“I was hiding *under* the table,” “I’m going *down* the slide,” “I’ll climb *up* the stairs”).

Questions for Reflection

1. How would you expand the castle project to include additional mathematical skills such as comparing, measuring, counting, and classifying?
2. What materials in your preschool environment engage children in exploring and manipulating shapes?
3. What songs or games involving movement in space do you sing and play with children? How could you use these opportunities to encourage children to use words describing spatial relationships?
4. How could you use hands-on construction activities (such as the Building a Castle project described above) to compare and discuss the attributes of shapes?
5. In what ways could you support and scaffold English learners’ access to learning English words for shapes and spatial relationships?

Mathematical Reasoning

Mathematical reasoning is a key process in learning and developing mathematical knowledge in all areas of mathematics, including number and operations, classification, patterning, measurement, and geometry. It involves the ability to think and reason logically, to apply mathematical knowledge in different problem-solving situations, and to come up with different solutions. Mathematical reasoning is natural to most young children as they explore the environment and make sense of the world around them. As illustrated through different examples in the previous sections (see examples of "Mathematical Reasoning in Action"), young children engage in mathematical reasoning and problem solving in their play and as they go about their daily activities. "Does every child have one cup?" "Do we both have the same number of shells?" "How many children are here today?" "How much did the sunflower grow?" "What blocks can we use instead of the long rectangle block?" "Do more children like red apples or green apples?" Different situations in the everyday environment call for spontaneous mathematical thinking. Young children are eager and enthusiastic to search for solutions and apply different strategies, especially when the context is familiar and meaningful, the question or problem is understandable and important to them, and they have some knowledge base related to the problem.[27] Effective teachers build on children's natural motivation for mathematical reasoning and problem solving. They promote children's learning of new and progressively more advanced mathematical challenges and support the development of mathematical vocabulary and language.

1.0 Promoting Mathematical Reasoning and Problem Solving

Teachers play a key role in identifying natural situations of mathematical reasoning throughout the day and turning them into teachable moments. Teachers also play a key role in initiating opportunities for children to reason mathematically. They can nurture, facilitate, and encourage preschool children's mathematical reasoning.

Mathematical Reasoning in Action: Picking up Shovels in the Sandbox

VIGNETTE

The children cleaned up the play yard before going back inside. The teacher, Ms. Denise, had noticed that not all the shovels were picked up from the sandbox. Ms. Denise asked for help saying, "We need all five shovels back in the box so our toys aren't lost. I see here only three. We need more shovels in the box. How many more shovels do we need?" The teacher had noticed that Ling Wa, one of the older preschool children in the group, was counting her fingers, trying to find out how many shovels were missing.

Ling Wa suddenly said, "Ms. Denise, we need two more."

Ms. Denise went further, asking, "Do you think we need two more shovels?" How did you figure that out?"

Ling Wa explained, "We have three. Then two more, we will have—one, two, three, four, five (Ling Wa was counting on her fingers)."

Ms. Denise said, "You are right. We need two more. Can everybody help us find two more shovels in the sandbox?"

TEACHABLE MOMENT

Ms. Denise, the teacher, identified the situation of picking up the shovels as an opportunity for arithmetic thinking and reasoning. She described the situation: "We need all five shovels back . . . I see here only three" and posed a question: "How many more shovels do we need?" She challenged the children to think and solve an arithmetic problem. Even when Ling Wa came up with the right answer, Ms. Denise went further and asked "How did you figure that out?" The teacher gave Ling Wa an opportunity to explain her reasoning. Ling Wa, like many other children in this group, very much enjoyed figuring out the answer to a simple addition and subtraction problem. Recently, she had started using her fingers in solving such problems. Representing numbers in the problem with fingers or other objects (e.g., shovels) makes arithmetic reasoning more concrete and meaningful for young children.

The following interactions and strategies facilitate preschool children's mathematical reasoning:

Identify and create opportunities for mathematical reasoning. Teachers can provide children with opportunities of mathematical reasoning, whether through spontaneous questioning and reasoning with children or through carefully planned experiences. Teachers may use everyday activities to initiate moments of mathematical reasoning. For example, in the "Picking up Shovels" vignette described above, the teacher identified a clean-up situation as an opportunity to engage children in reasoning with numbers. Similarly, in the "More Cups" example (page 245), the teacher engaged the child in mathematical reasoning while setting the table for lunch. Opportunities for mathematical reasoning come up while teachers observe children closely and listen to their ideas and thoughts. Teachers capitalize on these moments to facilitate mathematical concepts and encourage children to apply and explain their reasoning. In "Who Has More Cars?" (page 243), for example, the teacher had noticed two children spontaneously counting their cars to show they have more and turned it into a teachable moment of mathematical reasoning and problem solving. Similarly, in the example of "Playing at the Water Table" (page 274), Ms. Frances observed Sara at the water table filling up a cup with water and pouring it into a bigger container. The teacher turned it into a mathematical reasoning experience of estimation and measurement, asking, "How many more cups do you think it would take to fill it up?"

The teacher may also plan in advance activities or experiences to engage children in mathematical reasoning related to particular concepts. For example, in "Tracking the Growth of Sunflowers" (page 279), the teacher planned an experience to engage children in measuring and comparing the height of sunflowers over time. Similarly, in the "Bagel Shop" activity on page 256, the teacher created a real-life setting of a bakery to engage children in counting and arithmetic reasoning.

Pose meaningful questions and challenge preschool children's thinking. One effective way to encourage preschool children to think and reason mathematically is by asking them questions that promote investigation and inquiry and challenge them to think through a problem and come up with a solution (e.g., "How do you think we can find out who has more cars?" "What is the class's favorite kind of apple? How can we find out . . .?" "Which do you think is taller, the table or the tower?" "What would happen if . . .?" "What other way could we sort the leaves?" "I wonder why . . .?"). By simply asking questions and listening to answers, teachers help children learn to reason. Give children time to answer a question or to solve a problem. Listen attentively to their ideas. Children's answers reveal what they understand and will inform teachers about how to best support their reasoning. Illustrate for children that, in many cases, there are different ways to solve a problem and more than one answer is possible.

Support preschool children in reasoning mathematically. Children may need a clue, encouragement, or the teacher's modeling of a strategy for solving a problem. In the example "Who Has More Cars?" (page 243), the teacher suggested to the children, "Let's count together." In "Tracking the Growth of Sunflowers (on page 279),” the teacher helped children compare the height of two sunflowers by using a string. Teachers should think out loud with children, make comments, and describe what the child is doing: “So here you put the green apples and here the red apples, but what about these apples?” “Long rectangle, short rectangle, long rectangle, short rectangle . . . look at your fence. You have a pattern.” Encourage preschool children to express their thoughts and explain their reasoning to the teacher as well as to their peers (e.g., “How did you figure it out?” “‘Look at the castle you built. Can you tell me what it looks like?”). Listening to and conversing with children helps them articulate the meaning of mathematical concepts, introduces them to mathematical language, and gives value to knowing how to “do math.” By making mathematical thinking conscious, teachers will do more of it and will develop a keener awareness of children’s use of mathematical strategies and math language.

Bringing It All Together

Engaging Families

The following idea may help families to develop children's mathematical reasoning:

✔ **Encourage parents to engage children in mathematical reasoning.** When talking about children's mathematical development, parents often think of their children's ability to count, name shapes, or say simple number facts (e.g., two plus two is four). It is important to communicate to parents what we mean by *mathematical reasoning.* It is about children being able to think mathematically and explore different ways of solving problems. To promote children's mathematical reasoning, parents should recognize mathematics in daily events and interactions and turn them into mathematical learning experiences. They can ask questions related to everyday situations: *How many more chairs do we need around the table? How can we divide these carrots equally among the four of you? Can you estimate how many spoonfuls it will take for you to finish your bowl of cereal?* Parents should encourage children to think. They may think aloud with children, listen to children's thoughts and answers, model solutions, and guide them through the thinking process.

Questions for Reflection

1. Think about a recent experience in which children in your group were engaged in mathematical thinking and reasoning.
   - What strategies have you used to engage children in mathematical reasoning?
   - What do you think children liked most about this experience? What did you like most about this experience?
   - What you would have added or changed in that experience?
2. Do you have children in your group who, like Ling Wa, enjoy figuring out the answers to simple addition and subtraction problems? How did you or would you find out? What would you do to support children's growth in mathematics?
3. What experiences related to your group's current focus or topic of interest would you offer children to engage them in mathematical reasoning?
4. How would you challenge different children in your group to reason mathematically according to their individual developmental level? How could you make a mathematical reasoning activity progressively more challenging?

Concluding Thoughts

Young children have a natural interest, curiosity, and competence to explore and construct mathematical concepts. Mathematics is a way of thinking and organizing the world around us. It is a natural part of day-to-day activities and events. Mathematics in preschool is learned through children's play and exploration as in the blocks area or the sandbox, through everyday routines such as setting the table and cleaning up, and through participation in teacher-initiated activities. Some teacher-initiated activities are designed with a focus on math, and others may focus on art, movement, literacy, or science but present opportunities for math learning. When teachers recognize the potential for exposure to math in different situations, they can turn everyday occurrences into exciting and effective mathematics-learning experiences. Children are excited to explore the size or volume of objects, to discover and create patterns, to manipulate and build with shapes, to sort and classify objects, and to try to figure out "how many." Teachers get to experience with children the day-to-day excitement of learning and discovering math. This process is joyful for the children and for the teacher, who guides and challenges them in building mathematical concepts, skills, and language.

Map of the Foundations

Domain → Mathematics

Strand → Algebra and Functions (Classification and Patterning)*

	At around 48 months of age	*At around 60 months of age*
Age		
Substrand	**1.0 Children begin to sort and classify objects in their everyday environment.**	**1.0 Children expand their understanding of sorting and classifying objects in their everyday environment.**
Foundation	**1.1** Sort and classify objects by *one* attribute into two or more groups, with increasing accuracy.	**1.1** Sort and classify objects by *one or more* attributes, into two or more groups, with increasing accuracy (e.g., may sort first by one attribute and then by another attribute).†
Examples	**Examples** • Selects some red cars for himself and some green cars for his friend, leaving the rest of the cars unsorted. • Chooses the blue plates from a variety of plates to set the table in the kitchen play area. • Sorts through laundry in the basket and takes out all the socks. • Places all the square tiles in one bucket and all the round tiles in another bucket. • Attempts to arrange blocks by size and communicates, "I put all the big blocks here and all the small ones there."	**Examples** • Sorts the large blue beads into one container and the small red beads in another. • Puts black beans, red kidney beans, and pinto beans into separate bowls during a cooking activity. • Arranges blocks on the shelf according to shape. • Sorts a variety of animal photographs into two groups: those that fly and those that swim. • Sorts buttons first by size and then each subgroup by color into muffin tin cups.

Includes notes for children with disabilities →

* Throughout these mathematics foundations many examples describe the child manipulating objects. Children with motor impairments may need assistance from an adult or peer to manipulate objects in order to do things such as count, sort, compare, order, measure, create patterns, or solve problems. A child might also use adaptive materials (e.g., large manipulatives that are easy to grasp). Alternately, a child might demonstrate knowledge in these areas without directly manipulating objects. For example, a child might direct a peer or teacher to place several objects in order from smallest to largest. Children with visual impairments might be offered materials for counting, sorting, or problem solving that are easily distinguishable by touch. Their engagement is also facilitated by using containers, trays, and so forth that contain their materials and clearly define their work space.

† Attributes include, but are not limited to, size, shape, or color.

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Early Childhood: Where Learning Begins – Mathematics. 1999. U.S. Department of Education. The site features book titles http://www.ed.gov/pubs/EarlyMath/appendix.html

Early Childhood Mathematics: Promoting Good Beginnings. The National Association for the Education of Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM) present a joint statement. http://www.naeyc.org/about/positions/mathematics.asp

50 Multicultural Books Every Child Should Know. Madison: Cooperative Children's Book Center, School of Education, University of Wisconsin-Madison, http://www.education.wisc.edu/ccbc/books/detailListBooks.asp?idBookLists=42

Investigations in Number, Data, and Space is a mathematics curriculum for kindergarten up to grade five that includes a library with a series of papers by D. H. Clements on teaching mathematics to young children. http://investigations.terc.edu/library/bookpapers/your_childs_geometric.cfm

Issues related to math and literacy in *Young Children,* a journal of the National Association for the Education of Young Children (NAEYC). http://journal.naeyc.org/search/search.asp?printResults=1&page=1&sesID=1234477375896

Launching Into Literacy and Math, Madison Metropolitan School District, *Picture Books That Nurture Mathematical Thinking: Ages 3-5* and *Math Concepts in Spanish/Bilingual Books for Infants, Toddlers, and Preschoolers.* http://www.madison.k12.wi.us/tnl/lilm/additional_resources/good_books_preschool.html

Math Perspectives Teacher Development Center provides mathematics educators of preschool children to sixth graders with tools, strategies, and assessments that will ensure that all students are successful in the study of mathematics and can use mathematics to solve problems and think and reason mathematically. http://www.mathperspectives.com

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National Council of Teachers of Mathematics (NCTM). http://www.nctm.org

Principles and Standards for School Mathematics. The new math standards released by NCTM in 2000 include standards for preschool to grade two. http://standards.nctm.org

Reading Is Fundamental. *100 of the Decade's Best Multicultural Read-Alouds, Pre-kindergarten through Grade 8*, selected and annotated by J. Freeman. http://www.rif.org/educators/books/100_best_multicultural.mspx

Selected Book Pairs for Linking Math and Literacy by P. Whitin and D. J. Whitin. National Association for the Education of Young Children (NAEYC). http://journal.naeyc.org/btj/200503/06whitin.pdf

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Teaching Children Mathematics. The journal "offers activities, lesson ideas, teaching strategies, and problems through in-depth articles, departments, and features." "Early Childhood Corner" is a regular section of the journal. It features articles related to the teaching and learning of young children. http://my.nctm.org/eresources/journal_home.asp?journal_id=4

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Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

CCSS Mathematical Practices

OVERARCHING HABITS OF MIND
1. Make sense of problems and persevere in solving them
6. Attend to precision

REASONING AND EXPLAINING
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others

MODELING AND USING TOOLS
4. Model with mathematics
5. Use appropriate tools strategically

SEEING STRUCTURE AND GENERALIZING
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Source: McCallum, 2011

Handout 2.0.1

Standards for Mathematical Practice
MP 1 and MP 6

MP 1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP 6. Attend to precision

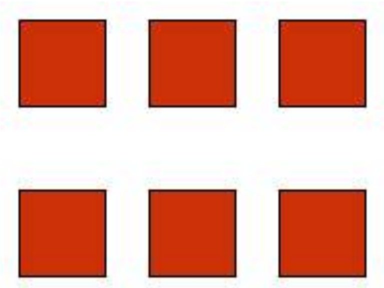
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they reach high school they have learned to examine claims and make explicit use of definitions.

Handout 2.1.3

Posing Questions and Responding to Students

Sample Question in Primary

Teacher: *Write a word sentence for the following picture:*



Student Response: *There are six in the picture.*

Teacher 1: *Right!*

Teacher 2: *Six what?*

Teacher 3: *Six what? Does anyone see something else? What else can you tell me about the picture?*

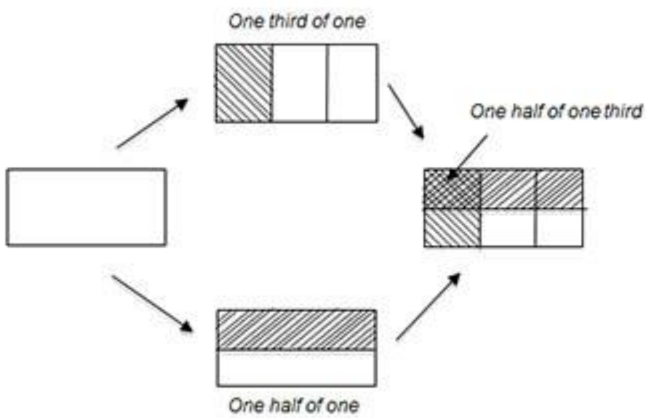
Handout 2.1.4

Posing Questions and Responding to Students

Sample Question in Upper Elementary

Teacher: *Students, draw a picture of one half of one third.*

Student Response: *Here is my picture:*



Teacher 1: *Right, the shaded part at the end shows one half of one third!*

Teacher 2: *How do you know that your last picture shows one half of one third?*

Teacher 3: *Explain your thinking. Why do you have the shape drawn four times? Convince me that the last one illustrates one half of one third.*

Handout 2.1.5

Posing Questions and Responding to Students

Sample Question in Middle School

Teacher: *Students, solve the equation 2(x - 4) + 3 = x - 4. Show all work.*

Student Response: *Here is my solution:*

$$2(x - 4) + 3 = x - 4$$
$$2x - 8 + 3 = x - 4$$
$$2x - 5 = x - 4$$
$$x = 1$$

Teacher 1: *Your solution is correct.*

Teacher 2: *How do you check to see if your answer is correct?*

Teacher 3: *Could you have done this problem another way? What could have been a different first step? Can you justify each step?*

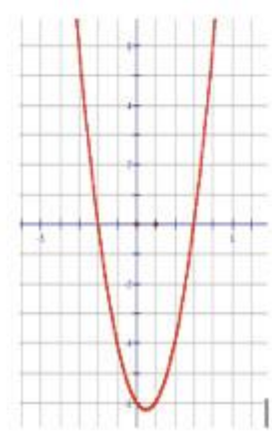
Handout 2.1.6

Posing Questions and Responding to Students

Sample Question in High School

Teacher: *Students, solve the inequality* $x^2 - x - 6 < 0$

Student Response: *I decided to do this by graphing. I graphed the function* $y = x^2 - x - 6$ *below:*



…So, the answer is that the inequality $x^2 - x - 6 < 0$ *occurs when* $-2 < x < 3$.

Teacher 1: *Where are your cases?*

Teacher 2: *This is correct. Why did you go with a graphing solution?*

Teacher 3: *If you graphed* $y = | x^2 - x - 6 |$, *what would that graph look like?*
How does that graph differ from the one you drew?
How is the answer connected to the zeros of $y = x^2 - x - 6$?

Handout 2.2.1

The Hook to Persevere

Primary Task Set

Task 1: Measure the length of your desk in centimeters.

Task 2: Measure the length of your desk using your hand span. Measure the length of the desk again using another student's hand span. Are the measures the same? Why might they be different?

Handout 2.2.2

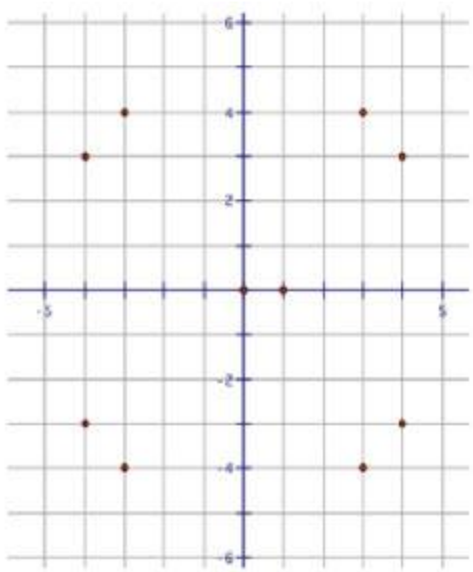
The Hook to Persevere

Upper Elementary Task Set

Task 1: On a coordinate grid, graph and label the following points:

(3, 4), (-3, 4), (4, -3), (-4, -3), (3, -4), (4, 3), (-3, -4), (-4, 3)

Task 2: Label the points drawn on the coordinate grid below. Connect the points to make a picture frame. Describe the line segments you created to make your frame.



Handout 2.2.3

The Hook to Persevere

Middle School Task Set

Task 1: Determine the volume of a rectangular prism that is 9 ft long, 12 feet wide and 15 inches deep. What would be the volume of a rectangular prism that was 1.5 times as long and 1.5 times as wide as the original prism?

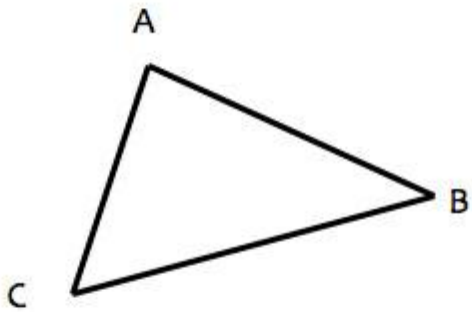
Task 2: The 7th graders at Sunview Middle School were helping to renovate a playground for the kindergartners at a nearby elementary school. City regulations require that the sand underneath the swings be at least 15 inches deep. The sand under both swing sets was only 12 inches deep when they started. The rectangular area under the small swing set measures 9 feet by 12 feet and required 40 bags of sand to increase the depth by 3 inches. How many bags of sand will the students need to cover the rectangular area under the large swing set if it is 1.5 times as long and 1.5 times as wide as the area under the small swing set?

Handout 2.2.4

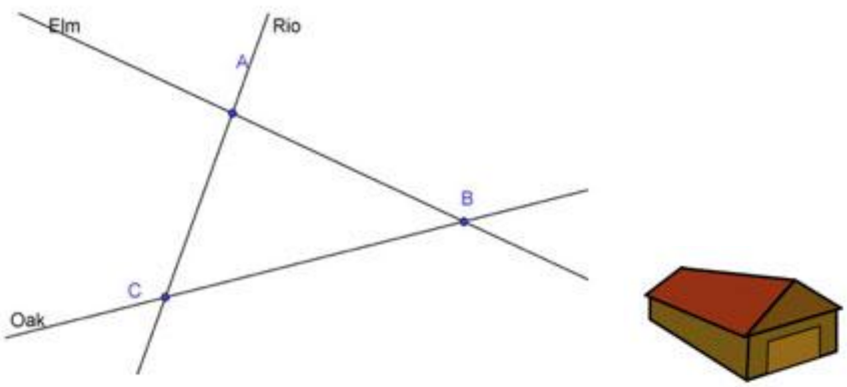
The Hook to Persevere

High School Task Set

Task 1: Locate the incenter of the following triangle:



Task 2: You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.



Handout 2.3.1

Focusing on Mathematical Statements

Read the statements below about rectangles (written by grade-level teachers of mathematics), and then answer the questions below.

Teacher Grade Level	Task: Write a statement about rectangles.
Special Education Elementary	If a figure has four right angles, then it is a rectangle.
7th	If a shape is a rectangle, then it is a parallelogram with at least two right angles.
5th	If a figure is a rectangle, then it is a parallelogram with at least one right angle.
4th	If a figure is a rectangle, then it is a four-sided polygon with four right angles.
2nd	If a figure is a rectangle, then it is a four-sided polygon with four right angles.
6th	If a parallelogram is a rectangle, it has four right angles.

Are the statements precise?

Are the stated conditions necessary?

Are the stated conditions sufficient?

Handout 3.0.1

Standards for Mathematical Practice
MP2 and MP3

MP2. Reasoning abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complimentary abilities to bear on problems involving quantitative relationships: the ability to decontextualize — to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents — and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP3. Construct viable arguments and critique the reasoning of others

Mathematically proficient students understand and use stated assumptions, definitions and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed and — if there is a flaw in an argument — explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Handout 3.0.2

Comparing Standards

CCSS for Mathematics Reasoning and Explaining Practices:

MP2. Reasoning Abstractly and Quantitatively

- *Make sense* of quantities and their relationships in problem situations
- *Decontextualize* — to abstract a given situation and represent it symbolically
- *Contextualize* — to probe into the referents for the symbols involved
- *Create* a coherent representation of the problem at hand
- *Consider* the units involved
- *Attend to* the meaning of quantities, not just how to compute them
- *Know* and flexibly *use* different properties of operations

MP3. Constructing Viable Arguments and Critiquing the Reasoning of Others

- *Use* stated assumptions, definitions, and previously established results in constructing arguments
- *Make* conjectures
- *Build* a logical progression of statements
- *Analyze* situations by breaking them into cases
- *Recognize and use* counterexamples
- *Justify* conclusions*, communicate the*m to others and *respond* to the arguments of others
- *Distinguish* correct logic or reasoning from that which is flawed

Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Adapted from the CCSS Initiative Web site at
http://www.corestandards.org/

Handout 3.0.2 (Cont.)

CCSS for English Language Arts Anchor Standards:

This is a partial list of the ELA Anchor Standards, organized in a different order than that presented in the CCSS.

Reading: Key Ideas and Details

1. Read closely to determine what the text says explicitly and to make logical inferences from it; cite evidence when writing or speaking to support conclusions drawn from the text.

Reading: Integration of Knowledge and Ideas

8. Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.

Writing: Text Types and Purposes

1. Write arguments to support claims in an analysis of substantive topics of texts, using valid reasoning and relevant and sufficient evidence.

Writing: Production and Distribution of Writing

5. Develop and strengthen writing as needed by planning, revising, editing, rewriting, or trying a new approach.

6. Use technology, including the Internet, to produce and publish writing and to interact and collaborate with others.

Writing: Research to Build and Present Knowledge

7. Conduct short as well as more sustained research projects based on focused questions, demonstrating understanding of the subject under investigation.

8. Gather relevant information from multiple print and digital sources, assess the credibility and accuracy of each source, and integrate the information while avoiding plagiarism.

9. Draw evidence from literary or informational texts to support analysis, reflection, and research.

Speaking and Listening: Comprehension and Collaboration

1. Prepare for and participate effectively in a range of conversations and collaborations with diverse partners, building on others' ideas and expressing their own clearly persuasively.

3. Evaluate a speaker's point of view, reasoning, and use of evidence and rhetoric.

Speaking and Listening: Comprehension and Collaboration

4. Present information, findings, and supporting evidence such that listeners can follow the line of reasoning and the organization, development, and style are appropriate to task, purpose, and audience.

Adapted from the CCR Anchor Standards for ELA on the CCSS Initiative Web page at
http://www.corestandards.org/ELA-Literacy/CCRA/R

Handout 3.1.3

Sample Strategies for Differentiating Instruction

Consider the recommendations below to engage all students as they develop their mathematics reasoning skills:

1. Presenting information in multiple ways
2. Ensuring that students understand the text of problems
3. Using multiple resources and modes
4. Understanding flexibility in ways students respond
5. Maintaining the high cognitive demand of tasks/rigorous content
6. Providing diverse avenues of action and expression
7. Using performance-based assessments
8. Using language as a resource for learning not only as a tool for communicating, but also as a tool for thinking and reasoning mathematically
9. Focusing on mathematical discourse and academic language
10. Asking students to participate in mathematical reasoning by making conjectures, presenting explanations, and/or constructing arguments
11. Encouraging English learners to produce explanations, presentations, and discussions as they learn English

Handout 3.2.1

Taxonomy of Questions in Mathematical Discourse: Questions and Responses

(adapted from work by Edith Prentice Mendez; Mitchell Nathan and Suyeon Kim)

Question Type	Description
External authority	Answers are attributed to someone else, teacher, parent, or "I just knew it".
Confirm – Is It?	Agree or disagree. Choose. Categorize.
Recall – What is it?	Knowledge produced from memory (e.g., facts, calculations, definitions).
Explain – What is true?	Produces new information. Gives solutions with enough clarity and detail to be understood. Gives examples. Convinces others.
Justification – Why is it true?	Provides evidence for and against the claim. Relates concepts to situations, new concepts, concept to question.
Generalizations – Is it always true?	Communicates reasoning about commonalities in patterns, procedures, structures, and relationships.

	Questions	Responses
External Authority	Any question.	Last year's teacher told me.
Confirm	Do you agree? Disagree? Is it …. or …..?	Yes / no Thumbs up / down
Recall	How many….? What did … say? What is…	It is … The answer is ….
Explain	How did you ..? Can you explain..? How are they the same? Different? And then what did you do?	First I … then I… and then… They are … because ….
Justify	Why would you…? Why does that work? Is there another way? What do you think?	If … then… because … If … so … because… So it would be …. because…
Generalize *(often combined with justification)*	Why? Does it always work? Is there a rule?	*Sentences will be followed with justifications.* What I'm noticing is … It will always…. Anytime ….

Handout for 3.2.3

Engaging Diverse Learners

Many students need additional supports when presented with the tasks of explaining and justifying. Consider the recommendations below to engage all students as they develop their mathematics reasoning skills:

English learners:

"Instructional recommendations for teaching the Common Core to English learners include a focus on English learner's mathematical reasoning; a shift to focus on mathematical discourse practices, moving away from simplified views of language, and support for English learners as they engage to complex mathematical language" (Moschkovich, 2012).

"Scaffolding should be provided through the use of multiple representations including choice of texts and tools (dictionaries, glossaries), teaching of key vocabulary, visual representations, models, strategic questioning, and coaching" (Santos, et al., 2012).

"Classroom environments that make ample use of the multiple modes — pictures, diagrams, presentations, written explanations, and gestures — afford English learners the means first to understand the mathematics with which they are engaged and second to express the thinking behind their reasoning and problem solving" (Driscoll, et. al. 2012).

Students receiving special education services:

"... strategies support student engagement by presenting information in multiple ways and allowing for students to access and express what they know in a variety of ways" (McNulty & Gloeckler, 2011).

"... a scientifically valid framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways that students respond or demonstrate knowledge and skills, and in the ways students are engaged" (CCSS Initiative, 2010).

Handout 4.2.1

Quotes on Modeling

Early Grades: *"Children naturally attempt to model the action or relationships in problems. They first directly model the situations or relationships with physical objects. Children's solution strategies are first fairly exact models of problem actions or relationships. Counting and Direct Modeling strategies are simply specific instances of the fundamental principle of modeling. It is helpful to think of these strategies as attempts to model problems rather than as a collection of distinct strategies. The conception of problem solving as modeling not only serves as a basis for understanding children's strategies for solving addition, subtraction, multiplication, and division problems, it also can provide a unifying framework for thinking about problem solving in the primary grades."*

Carpenter, et al., 1999

Middle Grades: *"The habit of problem posing, creating representations, explaining connections, and testing and checking are central to the development of interesting and new mathematics and applications. Real world applications often involve many variables, incomplete information, multiple methods of solution, and answers that vary according to the assumptions, and simplifications made and approach taken. Encounters with such settings dispel students' notion that the trademark of mathematics is the exactness and uniqueness of results. Rather recognition of underlying structure and abstraction become dominant features of the discipline. Teachers must help students become comfortable with uncertainty while striving for clarity in their descriptions and analyses. Students must accept that creativity and clear communication are part of active learning and discovery. Lastly, they must be curious and willing to take risks. Successful students in traditional math classes are rewarded for speed and technical accuracy. A different type of confidence is required when they begin posing problems with no immediate clear method of solution and no guarantee that a solution can be found".*

Abrams, 2001

High School: *"Mathematical modeling is a form of problem solving. A mathematical model should be mathematically accurate and portray a situation in the real world. Modeling considers solutions in terms of the situation that spawned the mathematical problem. A student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. At the high school level, teachers are expected to provide opportunities that allow students to examine how multiple representations support different ways of thinking about and manipulating mathematical objects. Students need an opportunity to practice converting among different representations for a given situation to create flexibility with modeling. These opportunities should emphasize selection of a certain representation for a mathematical situation based on what information the representation needs to convey."*

Christinson, et al., 2012

Handout 4.2.2

Examples of Grades K–2 Modeling Activities

The problems below represent Level 1 modeling tasks:

Problem 1: Connie had 5 marbles. Juan gave her 8 more marbles. How many marbles does Connie have altogether?

Problem 2: Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left?

Problem 3: Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?

Problem 4: Connie has 13 marbles. Juan has 5 marbles. How many more marbles does Connie have than Juan?

Problem 5: Gene has 4 tomato plants. There are 6 tomatoes on each plant. How many tomatoes are there altogether?

Problem 6: Ellen walks 3 miles an hour. How many hours will it take her to walk 15 miles?

Problem 7: Jan bought 7 pies. He spent a total of $28. If each pie cost the same amount, how much did one pie cost?

Problem 8: The giraffe in the zoo is 3 times as tall as the kangaroo. The kangaroo is 6 feet tall. How tall is the giraffe?

For additional practice, refer to *Children's Mathematics Cognitively Guided Instruction* (Carpenter, 1999), a publication that includes videos showing students engaging in these tasks.

Handout 4.2.3

Examples of Grades 3–5 Modeling

This problem represents a Level 2 modeling task.

Problem 1: Austin's mom is making healthy snacks for the students in his class. She finds a recipe on the Internet with the following ingredients. However, she didn't see how many servings the recipe will make. What are some reasonable estimates on how much ingredients Austin's mom needs to make enough for all the students in the class to get a healthy snack? What information would be helpful to know?

http://kidshealth.org/kid/recipes/recipes/snack_mix.html#cat20229

Prep time: 5 minutes

What you need:
–1 cup whole grain cereal (squares or Os work best)
–¼ cup dried fruit of your choice
–¼ cup nuts, such as walnut pieces, slivered almonds, or pistachios
–¼ cup small, whole-grain snack crackers or pretzels

Equipment and supplies:
–Large bowl
–Measuring cups
–Large spoon

What to do:
–Measure out ingredients
–Combine in large bowl

This problem represents a Level 3 modeling task:

Problem 2: Lamar wants a new toy truck that sells for $25. He has $3 now. Create a plan that would help Lamar buy his truck three weeks from today

Handout 4.3.1

Quotes on Tools

Grades K–2: *"Educational research indicates that the most valuable learning occurs when students actively construct their own mathematical understanding. One way to facilitate this is to provide opportunities for children to explore, develop, test, discuss, and apply ideas. Extensive and thoughtful use of physical materials, particularly in the primary grades, is conducive to the concrete kinds of learning that lay a satisfactory foundation for the development of this mathematical understanding"*

Johnson, 2012

Grades 3–5: *"Manipulative materials in teaching mathematics to students hold the promise that manipulatives will help students understand mathematics. At the same time, as with any "cure", manipulatives hold potential for harm if they are used poorly. (No matter what the grade level of the students, the sole use of manipulatives should not be for modeling procedures; instead manipulatives should be made available as tools for problem solving.) Manipulatives that are improperly used will convince students that two mathematical worlds exist — manipulative and symbolic. All mathematics comes from the real world. Then the real situation must be translated into the symbolism of mathematics for calculating. For example, putting three goats with five goats to get eight goats is the real world situation but on the mathematics level we say 3+5 = 8 (read three add five equals eight). These are not two different worlds but they are in the same world expressing the concepts in different ways."*

Teaching Today, 2012

Grades 6–12: "The effectiveness of hands-on learning does not end in 5th grade. Research indicates that students of all ages benefit by being introduced to mathematical concepts through physical exploration. Planning lessons that proceed from concrete to pictorial to abstract representations of concepts makes content mastery accessible to students of all ages. With concrete exploration (through touching, seeing, and doing), students can gain deeper and longer-lasting understandings of math concepts."

Teaching Today, 2012

Handout 4.4.1

Using Tools: K–2 Task and 3–5 Task

Grades K–2 Task: What's Missing?

Tools: Colored counters

Show the students six counters. Ask each student to close his/her eyes. Hide some of the counters under a sheet of heavy paper. When the student opens his/her eyes, s/he determines how many counters are hidden based upon the number of counters still showing.

Grades 3–5 Task: Battle Ship Using Grid Paper

Tools
Grid paper and colored pencils; one color for the ships and another for explosions on their ships and their enemy's ships. This is how they will keep track of what ordered pairs have been called.

Setup
Students begin by folding the grid paper in half. They need to draw coordinate axes on both the top half and the bottom half and label the *x*- and *y*-axes with the numbers 1–10 on each axis. The students will need to illustrate 5 ships on ordered pairs and label the ordered pairs. They should draw:

Two ships sitting on 2 ordered pairs
One ship sitting on 3 ordered pairs
One ship sitting on 4 ordered pairs
One ship sitting on 5 ordered pairs

Remind the students that the bottom half of the grid paper contains their boats (or "Navy") and the top half has their opponent's boats.

Actions
Students play in pairs sitting opposite each other and take turns calling out ordered pairs. Players should keep a list of the ordered pairs they call out written in (x,y) form on a piece of paper that both players can see. This will ensure there is no disagreement about what has been called (it is common for students to transpose the coordinates).

Then they are to mark the ordered pair they call out on the top coordinate plane. They should mark in black if they missed and red if they hit their opponent's boat. On the bottom half of the grid paper they are to color black for the ordered pairs their opponent calls out and color red for the ordered pairs that hit their ship.

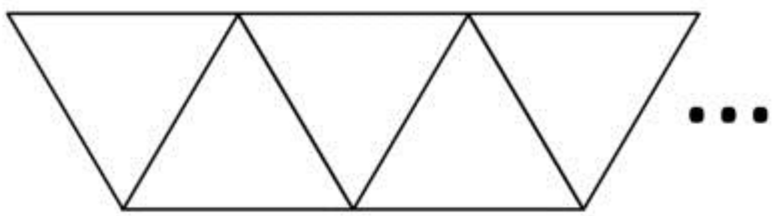
Handout 4.4.2

Using Tools: Middle School

Middle School Task: Triangular Tables

Tools
Pattern blocks

A classroom has triangular tables. There is enough space at each side of a table to seat one student. The tables in the class are arranged in a row (as shown in the picture below).



How many students can sit around one table? Around a row of two tables? Around a row of three tables?

Find an algebraic expression that describes the number of students that can sit around a row of *n* tables. Explain in words how you found your expression.

If you could make a row of 125 tables, how many students would be able to sit around it?

If there are 26 students in the class, how many tables will the teacher need to seat all of them around a row of tables?

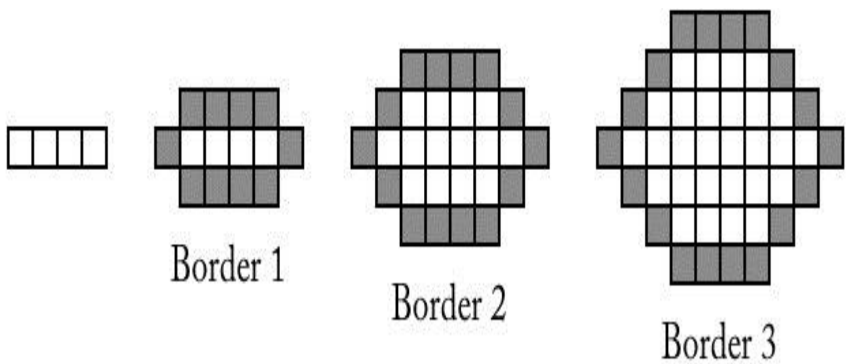
Handout 4.4.3

Using Tools: High School

High School Task: Kitchen Floor Tiles

Tools
Colored Tiles

Fred decides to cover the kitchen floor with tiles of different colors. He starts with a row of four tiles of the same color. He surrounds these four tiles with a border of tiles of a different color (Border 1). The design continues as shown below:



Dina writes, $t = 4\,(b-1) + 10$ where t is the number of tiles in each border and b is the border number. Explain why Dina's equation is correct.

Emma wants to start with five tiles in a row. She reasons, “Dina started with four tiles and her equation was $t = 4(b - 1) + 10$. So if I start with five tiles, the equation will be $t = 5(b - 1) + 10$. Is Emma’s statement correct? Explain your reasoning.

If Emma starts with a row of n tiles, what should the formula be?

Handout 5.0.1

Standards for Mathematical Practice
MP7 and MP8

MP7. Look for and make use of structure

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7\times5+7\times3$ in preparation for learning about the distributive property. In the expression $x2+9x+14$, older students can see the 14 as 2×7 and the 9 as $2+7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5-3(x-y)2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

MP8. Look for and express regularity in repeated reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1,2) with slope 3, middle school students might abstract the equation $(y-2)/(x-1)=3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x2+x+1)$, and $(x-1)(x3+x2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Handout 5.1

Consecutive Sums

Some numbers can be written as a sum of consecutive positive integers:

$$6 = 1 + 2 + 3$$
$$15 = 4 + 5 + 6$$
$$= 1 + 2 + 3 + 4 + 5$$

Which numbers have this property? Explain.

Let's look at what might be expected of students at each grade span when working on this problem.

K–2 Example

Write the first 10 numbers as a sum of other numbers. Which of these sums contain only consecutive numbers?

3–5 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 15 &= 4 + 5 + 6 \\ &= 1 + 2 + 3 + 4 + 5 \end{aligned}$$

In small groups, find all numbers from 1–100 that can be written as a consecutive sum. Look for patterns as you work. Conjecture which numbers can and which cannot be expressed as a consecutive sum. How can some of the sums be used to find others?

Handout 5.1

Consecutive Sums (cont.)

6–8 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

$$\begin{aligned} 6 &= 1+2+3 \\ 15 &= 4+5+6 \\ &= 1+2+3+4+5 \end{aligned}$$

Which numbers have this property?

Sally made this conjecture: “Powers of 2 cannot be expressed as a consecutive sum.”

Agree or disagree and explain your reasoning.

9–12 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

$$\begin{aligned} 6 &= 1+2+3 \\ 15 &= 4+5+6 \\ &= 1+2+3+4+5 \end{aligned}$$

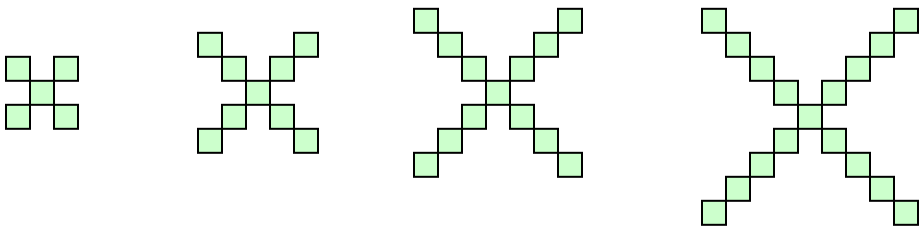
Exactly which numbers have this property?

When investigating this problem, Joe made the following conjecture: “A number with an odd factor can be written as a consecutive sum and the odd factor will be the same as the number of terms.” Agree or disagree with this statement and explain your reasoning.

Handout 5.2.1

Square Tiles

Tiles are arranged to form pictures like the ones below:



Picture 1 Picture 2 Picture 3 Picture 4

A. Find a direct formula that enables you to calculate the number of square tiles in Picture "n." How did you obtain your formula?

   If the solution has been obtained numerically, is there a way to explain your formula from the figures?

B. How many squares will there be in Picture 75? Explain.

C. Can you think of another way of finding a direct formula?

D. Two 6th graders came up with the following two formulas:

   Kevin's direct formula is: T = (n x 2) + (n x 2) + 1, where "n" means picture number and "T" means total number of squares.

   Is his formula correct? Why or why not?

E. Melanie's direct formula is: T = (n x 2) + 1 + (n x 2) + 1 –1, where "n" and "T" mean the same thing as in Kevin's formula.

   Is her formula correct? Why or why not?

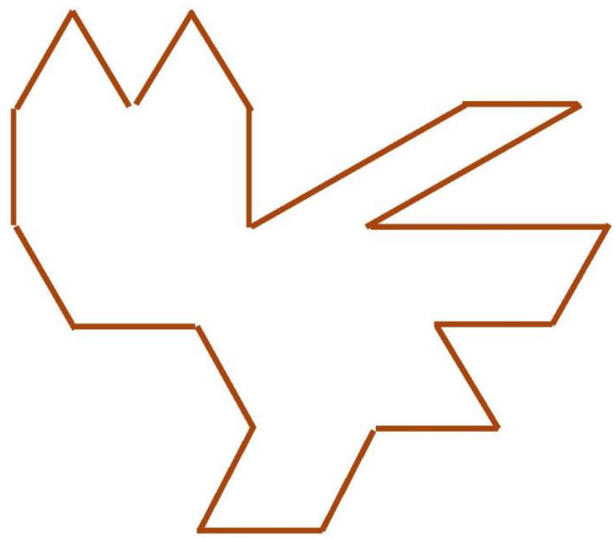
F. Which formula is correct: Kevin's formula, Melanie's formula, or your formula? Explain.

Handout 5.3.1

K–2 Geometry Example

Consider what might make the task below easier for students in K–2 classrooms. What questions would you ask to reinforce the relationships among the pattern block pieces?

Brian made the animal picture below out of pattern blocks. Can you show how Brian made his animal figure using pattern blocks?



3–5 Geometry Example

While you work on the problem below, think about what questions you might ask students in grades 3–5 to assist them in finding the solution.

Two vertices of a triangle are located at (4,0) and (8,0). The perimeter of the triangle is 12 units. What are all possible locations for the third vertex? How do you know you have them all?

Source: Driscoll, 2007

Handout 5.3.2

6–8 Geometry Example

View the first 10 minutes of the Third International Math and Science Study (TIMSS) video on changing shape without changing area (you can view the entire clip after you complete the activity below):

http://timssvideo.com/67

Construct the land figure with the crooked boundary posed by the teacher to the class in the video above. Find a way to make the boundary straight without changing the area.

9–12 Geometry Example

Explore the proof of the Pythagorean Theorem presented in the dynamic NTCM link below:

http://illuminations.nctm.org/ActivityDetail.aspx?id=30

Consider why this proof of the Pythagorean Theorem works and where structure plays a role in the proof.

The Eight Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them**
   Making sense and persevering are habits of mind needed by all students to be successful learners of mathematics. Before a student can engage in mathematics, they need to make sense of what they are being asked to consider.

2. **Reason abstractly and quantitatively**
   Reasoning abstractly requires that students make sense of quantities and their relationships in problem situations. Students decontextualize and contextualize mathematics; they translate problem situations into symbols which they are able to manipulate and, as they manipulate the symbols, refer back to the problem situation to make sense of their work.

3. **Construct viable arguments and critique the reasoning of others**
   Constructing arguments requires that students use stated assumptions, definitions, and previous results. They make conjectures, justify their conclusions, and communicate them to others. They respond to the arguments of others.

4. **Model with mathematics**
   Modeling with mathematics requires that students make assumptions and approximations to simplify a situation, realizing these may need revision later, and that students interpret mathematical results in the context of the situation and reflect on whether they make sense.

5. **Use appropriate tools strategically**
   Using tools strategically requires that students are familiar with appropriate tools to decide when each tool is helpful, know both benefits and limitations, detect possible errors, and identify relevant external mathematical resources and use them to pose or solve problems.

6. **Attend to precision**
   Precision refers to the accuracy with which students use mathematical language and symbols as well as precision in measurement.

7. **Look for and make use of structure**
   Looking for structure refers to students' understanding and using properties of number systems, geometric features and relationships, and patterns of a variety of types to solve problems.

8. **Look for and express regularity in repeated reasoning**
   Looking for regularity in repeated reasoning refers to the process of noticing repeated patterns or attributes and using those to abstract and express general methods, expressions or equations, or relationships.

Common Core State Standards, Standards for Mathematical Practice
Questions to Facilitate Student Thinking and Learning

Make sense of problems and persevere in solving them.

- **Make Sense of the Problem** *Can students understand, define, formulate, or explain the problem or task?*
  - What is this problem about?
  - What do you need to find out?
  - What information do you have?
  - Is there something that can be eliminated or that is missing?
  - What assumptions do you have to make?
- **Make Conjectures** *Can students describe the meaning of the solution and plan a solution pathway?*
  - What would be a good (estimate/prediction)?
  - What are some possible…(next steps/strategies/solutions)?
  - What patterns do you see?
  - What would happen if…? What if not?
  - What might be an easier problem you could start with?
- **Monitor and Evaluate Progress** *Can students vary the approach if one approach is not working?*
  - Would another method work as well or better?
  - Is there another way to (draw, explain, say…) that?
  - What did not work? Why?
  - Have you tried…(tables, lists, diagrams, manipulatives…)?
  - Would it help to draw a diagram or make a sketch?
- **Assess Solutions** *Do students evaluate the accuracy and meaning of their solution?*
  - Other than retracing your steps, how can you determine if your answer is correct?
  - Is the solution reasonable, considering the context?
  - Is that the only possible solution?
  - How do you know if you are finished?
  - Does that make sense?

Reason abstractly and quantitatively.

- **Make Sense of Quantities and Relationships** *Do students see relationships and recognize central ideas?*
  - What does this number represent?
  - What units are you working with?
  - What is the relationship between…?
- **Represent the Problem Symbolically** *Can students decontextualize and contextualize the quantities in relation to the problem situation?*
  - How can you represent this a different way?
  - What does this (symbol, diagram,…) represent?

Construct viable arguments and critique the reasoning of others.

- **Make Conjectures** *Do students use what they know to construct arguments?*
  - What you know about ___? How can you use that information?
  - What assumptions are you making?
  - Can you think of a counterexample?
  - Will that be true for all cases?
- **Justify Conclusions** *Can students articulate their thought processes? Do they explain their reasoning?*
  - How did you reach that conclusion?
  - How did you get your answer?
  - Why does that make sense?
- **Critique the Reasoning of Others** *Can students make sense of someone else's reasoning? Can they recognize correct reasoning from that which is flawed?*
  - What do you think about what ___ said?
  - Do you agree? Why or why not?
  - What questions does this raise for you?
  - What could you add to improve this line of reasoning?
  - Which is the most efficient method? Why?

Common Core State Standards, Standards for Mathematical Practice
Questions to Facilitate Student Thinking and Learning

Model with mathematics.

- **Apply the Mathematics** *Do students use what they know to solve problems arising in everyday life, society, and the workplace?*
  - How can you apply what you know to this problem?
  - What is the relationship between ___? How could you model this using…(a diagram, tables, graph, flowchart, equation)?
  - How could you (organize/represent) these ideas?
  - If this is not working for you, how might you adjust the (diagram, table, graph, equation,..) to represent the situation better?
- **Interpret Results** *Do students interpret their results in the context of the situation?*
  - Does that make sense in terms of the situation?
  - Does your answer need to be adjusted to fit the situation?

Use appropriate tools strategically.

- **Choose Appropriate Tools** *Can students use a variety of tools (paper and pencil, concrete models, rulers, protractors, compasses, calculators, and other technology) to solve problems?*
  - What could you use to help you ….?
  - What would be the (quickest/easiest/most efficient) way to …?
  - How could you use (calculator/computer) to…?
  - What are some (benefits/limitations) of (using/not using)…?

Attend to precision.

- **Communicate Precisely** *Do students use appropriate terminology and notation?*
  - How would you say that using appropriate math vocabulary?
  - What does that (symbol/variable/axis/number) represent?
  - Did you define your variables?
  - Did you label your (answer/diagram/graph) appropriately?
- **Express Answers Precisely** *Do students express numerical answers with a degree of precision appropriate for the problem context?*
  - Would it make sense to round your answer? To what place value? Why?
  - How exact does your answer need to be? (in terms of units or place value)

Look for and make use of structure.

- **Look for Structure** *Do students look closely to discern a pattern or structure?*
  - What do you notice about …?
  - How is this similar to …?
  - What are two different ways we could (look at/describe/write) at that?
- **Make Use of Structure** *Do students use patterns or structures they know to solve problems a different way or more efficiently?*
  - How could you use that (pattern/structure) to …?
  - What do you know about the value of this expression?

Look for and express regularity in repeated reasoning.

- **Look for Repeated Reasoning** *Can students recognize when calculations are repeated?*
  - What do you notice about the last few…?
  - How are these similar to each other?
- **Express Regularity** *Can students generalize methods?*
  - How could you describe that pattern in words?
  - How could you describe that pattern as a rule?

Adapted from Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions, NCTM, 1991

Common Core State Standards, Mathematics Practices

Questions for Planning and Observation

Make sense of problem and persevere in solving.
Do students:

Unpack the problem?
- What is the story?
- What are the given quantities?
- What needs to be found out?

Use strategies to enter the problem?
- Previous similar or simpler problems.
- Knows representations/models that work.
- Language needed to understand problem.

Recognize relationships in the problem? Relationships needed to find a solution?
- Solve a part of problem needed to solve second part?
- Relationship between quantities?
- Look for regularities, trends?

Know what the answer tells you? What form should the answer be? What units are called for?

Do strategies and results make senses?
- Can students explain why they are trying a particular strategy?
- If not making progress, can students change course and try a different strategy?
- Use another strategy to verify and explain solution?

What mathematics should be evident in all solutions? How will students see the same mathematics in each representation and solution?

Reason abstractly and quantitatively.
Do students:

Mathematize the problem?
- What are the given quantities?
- How do they relate to each other?

Represent the problem symbolically?
- Can students explain what symbols mean and how they relate to quantities, other symbols, representations, models?
- Explain the context of problem?

What properties and reasoning will support solutions? How can the problem be decomposed and recombined?

What are the units needed while solving and reporting answer?

Construct viable arguments and critique the reasoning others.
Do students:

Make conjectures?
- Explore the problem to support or disprove their conjecture?
- Refine or change their conjecture?

Construct their justification? Use objects? Drawings? Diagrams? Examples and counter examples? Cases?

Have opportunities to explain their conclusions and communicate their reasoning with others? What language is needed?

Have opportunities to ask useful questions to seek clarity? Follow the arguments of others looking for flaws and explaining them?

Questions for Planning and Observation, continued

Model with mathematics.
Do students:
Apply the mathematics to the problems?
Make and recognize assumptions and approximations?
Understand they may need to make revisions?
Identify important quantities and the relationships between them?
Interpret the mathematics in the context of the problem?
Reflect on the results?
- Make sense of solutions?
- Evaluate model to see if it can be improved?

Use appropriate tools strategically.
Do students:
Choose tools to fit the problem and know how to use them?
Recognize usefulness and limitations of tool?
Use technological tools to explore and deepen understanding?

Attend to precision.
Do students:
Communicate precisely to others?
- Do they use clear definitions?
- State the meaning of the symbols they use?

Calculate accurately and precisely?
Examine their claims and check reasoning?

Look for and make use of structure.
Do students:
Recognize the structure of the problem?
- Patterns (e.g., commutative property)
- Definitions (e.g., rectangles have 4 sides)
- Utilize properties
- Decompose and recombine numbers and expressions?

Are students able to shift perspective?

Look for and express regularity in repeated reasoning.
Do students:
Notice if calculations repeat themselves?
Look for general methods? Shortcuts?
Maintain oversight of process and attend to details?
Evaluate the reasonableness of the results?

North Bay Mathematics Project
Ben Ford, SSU, Faculty Advisor
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K, Counting and Cardinality
K–5, Operations and Algebraic Thinking

Progressions for the Common Core State Standards in Mathematics (draft)

29 May 2011

K, Counting and Cardinality; K–5, Operations and Algebraic Thinking

Counting and Cardinality and Operations and Algebraic Thinking are about understanding and using numbers. Counting and Cardinality underlies Operations and Algebraic Thinking as well as Number and Operations in Base Ten. It begins with early counting and telling how many in one group of objects. Addition, subtraction, multiplication, and division grow from these early roots. From its very beginnings, this Progression involves important ideas that are neither trivial nor obvious; these ideas need to be taught, in ways that are interesting and engaging to young students.

The Progression in Operations and Algebraic Thinking deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of the Progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measures, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers.

The generality of the concepts involved in Operations and Algebraic Thinking means that students' work in this area should be designed to help them extend arithmetic beyond whole numbers (see the NF and NBT Progressions) and understand and apply expressions and equations in later grades (see the EE Progression).

Addition and subtraction are the first operations studied. Initially, the meaning of addition is separate from the meaning of subtraction, and students build relationships between addition and subtraction over time. Subtraction comes to be understood as reversing the actions involved in addition and as finding an unknown addend. Likewise, the meaning of multiplication is initially separate from the meaning of division, and students gradually perceive relationships between division and multiplication analogous to those between addition and subtraction, understanding division as reversing the actions involved in multiplication and finding an unknown product.

Over time, students build their understanding of the properties of arithmetic: commutativity and associativity of addition and multiplication, and distributivity of multiplication over addition. Initially, they build intuitive understandings of these properties, and they use these intuitive understandings in strategies to solve real-world and mathematical problems. Later, these understandings become more explicit and allow students to extend operations into the system of rational numbers.

As the meanings and properties of operations develop, students develop computational methods in tandem. The OA Progression in Kindergarten and Grade 1 describes this development for single-digit addition and subtraction, culminating in methods that rely on properties of operations. The NBT Progression describes how these methods combine with place value reasoning to extend computation to multi-digit numbers. The NF Progression describes how the meanings of operations combine with fraction concepts to extend computation to fractions.

Students engage in the Standards for Mathematical Practice in grade-appropriate ways from Kindergarten to Grade 5. Pervasive classroom use of these mathematical practices in each grade affords students opportunities to develop understanding of operations and algebraic thinking.

4

Counting and Cardinality

Several progressions originate in knowing number names and the count sequence:[K.CC.1]

K.CC.1 Count to 100 by ones and by tens.

From saying the counting words to counting out objects Students usually know or can learn to say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects. Students become fluent in saying the count sequence so that they have enough attention to focus on the pairings involved in counting objects. To count a group of objects, they pair each word said with one object.[K.CC.4a] This is usually facilitated by an indicating act (such as pointing to objects or moving them) that keeps each word said in time paired to one and only one object located in space. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays (they need to ensure they reach every row or column and do not repeat rows or columns); circles (they need to stop just before the object they started with); and scattered configurations (they need to make a single path through all of the objects).[K.CC.5] Later, students can count out a given number of objects,[K.CC.5] which is more difficult than just counting that many objects, because counting must be fluent enough for the student to have enough attention to remember the number of objects that is being counted out.

K.CC.4a Understand the relationship between numbers and quantities; connect counting to cardinality.

a When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

From subitizing to single-digit arithmetic fluency Students come to quickly recognize the cardinalities of small groups without having to count the objects; this is called *perceptual subitizing*. Perceptual subitizing develops into *conceptual subitizing*—recognizing that a collection of objects is composed of two subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying "four"). Use of conceptual subitizing in adding and subtracting small numbers progresses to supporting steps of more advanced methods for adding, subtracting, multiplying, and dividing single-digit numbers (in several OA standards from Grade 1 to 3 that culminate in single-digit fluency).

From counting to counting on Students understand that the last number name said in counting tells the number of objects counted.[K.CC.4b] Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the

K.CC.4b Understand the relationship between numbers and quantities; connect counting to cardinality.

b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

more advanced counting-on methods in which a counting word represents a group of objects that are added or subtracted and addends become embedded within the total[1.OA.6] (see page 14). Being able to count forward, beginning from a given number within the known sequence,[K.CC.2] is a prerequisite for such counting on. Finally, understanding that each successive number name refers to a quantity that is one larger[K.CC.4c] is the conceptual start for Grade 1 counting on. Prior to reaching this understanding, a student might have to recount entirely a collection of known cardinality to which a single object has been added.

From spoken number words to written base-ten numerals to base-ten system understanding The NBT Progression discusses the special role of 10 and the difficulties that English speakers face because the base-ten structure is not evident in all the English number words.

From comparison by matching to comparison by numbers to comparison involving adding and subtracting The standards about comparing numbers[K.CC.6,K.CC.7] focus on students identifying which of two groups has more than (or fewer than, or the same amount as) the other. Students first learn to match the objects in the two groups to see if there are any extra and then to count the objects in each group and use their knowledge of the count sequence to decide which number is greater than the other (the number farther along in the count sequence). Students learn that even if one group looks as if it has more objects (e.g., has some extra sticking out), matching or counting may reveal a different result. Comparing numbers progresses in Grade 1 to adding and subtracting in comparing situations (finding out "how many more" or "how many less"[1.OA.1] and not just "which is more" or "which is less").

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Operations and Algebraic Thinking

Overview of Grades K–2

Students develop meanings for addition and subtraction as they encounter problem situations in Kindergarten, and they extend these meanings as they encounter increasingly difficult problem situations in Grade 1. They represent these problems in increasingly sophisticated ways. And they learn and use increasingly sophisticated computation methods to find answers. In each grade, the situations, representations, and methods are calibrated to be coherent and to foster growth from one grade to the next.

The main addition and subtraction situations students work with are listed in Table 1. The computation methods they learn to use are summarized in the margin and described in more detail in the Appendix.

Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away. Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2. Counting On. Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Level 3. Convert to an Easier Problem. Decompose an addend and compose a part with another addend.

See Appendix for examples and further details.

7

Table 1: Addition and subtraction situations

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	**Both Addends Unknown**[1]	**Addend Unknown**[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

In each type (shown as a row), any one of the three quantities in the situation can be unknown, leading to the subtypes shown in each cell of the table. The table also shows some important language variants which, while mathematically the same, require separate attention. Other descriptions of the situations may use somewhat different names. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

Kindergarten

Students act out adding and subtracting situations by representing quantities in the situation with objects, their fingers, and math drawings (MP5).K.OA.1 To do this, students must mathematize a real-world situation (MP4), focusing on the quantities and their relationships rather than non-mathematical aspects of the situation. Situations can be acted out and/or presented with pictures or words. Math drawings facilitate reflection and discussion because they remain after the problem is solved. These concrete methods that show all of the objects are called Level 1 methods.

Students learn and use mathematical and non-mathematical language, especially when they make up problems and explain their representation and solution. The teacher can write expressions (e.g., $3-1$) to represent operations, as well as writing equations that represent the whole situation before the solution (e.g., $3-1=\square$) or after (e.g., $3-1=2$). Expressions like $3-1$ or $2+1$ show the operation, and it is helpful for students to have experience just with the expression so they can conceptually chunk this part of an equation.

K.OA.1 Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

Working within 5 Students work with small numbers first, though many kindergarteners will enter school having learned parts of the Kindergarten standards at home or at a preschool program. Focusing attention on small groups in adding and subtracting situations can help students move from perceptual subitizing to conceptual subitizing in which they see and say the addends and the total,• e.g., "Two and one make three."

• Note on vocabulary: The term "total" is used here instead of the term "sum." "Sum" sounds the same as "some," but has the opposite meaning. "Some" is used to describe problem situations with one or both addends unknown, so it is better in the earlier grades to use "total" rather than "sum." Formal vocabulary for subtraction ("minuend" and "subtrahend") is not needed for Kindergarten, Grade 1, and Grade 2, and may inhibit students seeing and discussing relationships between addition and subtraction. At these grades, the terms "total" and "addend" are sufficient for classroom discussion.

Students will generally use fingers for keeping track of addends and parts of addends for the Level 2 and 3 methods used in later grades, so it is important that students in Kindergarten develop rapid visual and kinesthetic recognition of numbers to 5 on their fingers. Students may bring from home different ways to show numbers with their fingers and to raise (or lower) them when counting. The three major ways used around the world are starting with the thumb, the little finger, or the pointing finger (ending with the thumb in the latter two cases). Each way has advantages physically or mathematically, so students can use whatever is familiar to them. The teacher can use the range of methods present in the classroom, and these methods can be compared by students to expand their understanding of numbers. Using fingers is not a concern unless it remains at the first level of direct modeling in later grades.

Students in Kindergarten work with the following types of addition and subtraction situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown (see the dark shaded types in Table 2). Add To/Take From situations are action-oriented; they show changes from an initial state to a final state. These situations are readily modeled by equations because each aspect of the situation has a representation as number, operation ($+$ or $-$), or equal sign (=, here with the meaning of "becomes," rather than the more general "equals").

Table 2: Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	**Both Addends Unknown[1]**	**Addend Unknown[2]**
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Darker shading indicates the four Kindergarten problem subtypes. Grade 1 and 2 students work with all subtypes and variants. Unshaded (white) problems are the four difficult subtypes or variants that students should work with in Grade 1 but need not master until Grade 2. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

In Put Together/Take Apart situations, two quantities jointly compose a third quantity (the total), or a quantity can be decomposed into two quantities (the addends). This composition/decomposition may be physical or conceptual. These situations are acted out with objects initially and later children begin to move to conceptual mental actions of shifting between seeing the addends and seeing the total (e.g., seeing children or seeing boys and girls, or seeing red and green apples or all the apples).

The relationship between addition and subtraction in the Add To/Take From and the Put Together/Take Apart action situations is that of reversibility of actions: an Add To situation undoes a Take From situation and vice versa and a composition (Put Together) undoes a decomposition (Take Apart) and vice versa.

Put Together/Take Apart situations with Both Addends Unknown play an important role in Kindergarten because they allow students to explore various compositions that make each number.[K.OA.3] This will help students to build the Level 2 embedded number representations used to solve more advanced problem subtypes. As students decompose a given number to find all of the partners• that compose the number, the teacher can record each decomposition with an equation such as $5 = 4 + 1$, showing the total on the left and the two addends on the right.• Students can find patterns in all of the decompositions of a given number and eventually summarize these patterns for several numbers.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4 + 1$).

• The two addends that make a total can also be called partners in Kindergarten and Grade 1 to help children understand that they are the two numbers that go together to make the total.

• For each total, two equations involving 0 can be written, e.g., $5 = 5 + 0$ and $5 = 0 + 5$. Once students are aware that such equations can be written, practice in decomposing is best done without such 0 cases.

Equations with one number on the left and an operation on the right (e.g., $5 = 2 + 3$ to record a group of 5 things decomposed as a group of 2 things and a group of 3 things) allow students to understand equations as showing in various ways that the quantities on both sides have the same value.[MP6]

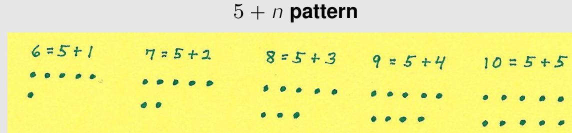
MP6 Working toward "using the equal sign consistently and appropriately."

Working within 10 Students expand their work in addition and subtraction from within 5 to within 10. They use the Level 1 methods developed for smaller totals as they represent and solve problems with objects, their fingers, and math drawings. Patterns such as "adding one is just the next counting word"[K.CC.4c] and "adding zero gives the same number" become more visible and useful for all of the numbers from 1 to 9. Patterns such as the $5 + n$ pattern used widely around the world play an important role in learning particular additions and subtractions, and later as patterns in steps in the Level 2 and 3 methods. Fingers can be used to show the same 5-patterns, but students should be asked to explain these relationships explicitly because they may not be obvious to all students.[MP3] As the school year progresses, students internalize their external representations and solution actions, and mental images become important in problem representation and solution.

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

$5 + n$ **pattern**



MP3 Students explain their conclusions to others.

Student drawings show the relationships in addition and subtraction situations for larger numbers (6 to 9) in various ways, such as groupings, things crossed out, numbers labeling parts or totals, and letters or words labeling aspects of the situation. The symbols $+$, $-$, or $=$ may be in the drawing. Students should be encouraged to explain their drawings and discuss how different drawings are the same and different.[MP1]

Later in the year, students solve addition and subtraction equations for numbers within 5, for example, $2 + 1 = \square$ or $3 - 1 = \square$, while still connecting these equations to situations verbally or with drawings. Experience with decompositions of numbers and with Add To and Take From situations enables students to begin to fluently add and subtract within 5.[K.OA.5]

Finally, composing and decomposing numbers from 11 to 19 into ten ones and some further ones builds from all this work.[K.NBT.1] This is a vital first step kindergarteners must take toward understanding base-ten notation for numbers greater than 9. (See the NBT Progression.)

The Kindergarten standards can be stated succinctly, but they represent a great deal of focused and rich interactions in the classroom. This is necessary in order to enable all students to understand all of the numbers and concepts involved. Students who enter Kindergarten without knowledge of small numbers or of counting to ten will require extra teaching time in Kindergarten to meet the standards. Such time and support are vital for enabling all students to master the Grade 1 standards in Grade 1.

MP1 Understand the approaches of others and identify correspondences

K.OA.5 Fluently add and subtract within 5.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Grade 1

Students extend their work in three major and interrelated ways, by:

- Representing and solving a new type of problem situation (Compare);
- Representing and solving the subtypes for all unknowns in all three types;
- Using Level 2 and Level 3 methods to extend addition and subtraction problem solving beyond 10, to problems within 20. In particular, the OA progression in Grade 1 deals with adding two single-digit addends, and related subtractions.•

• Other Grade 1 problems within 20, such as $14 + 5$, are best viewed in the context of place value, i.e., associated with 1.NBT.4. See the NBT Progression.

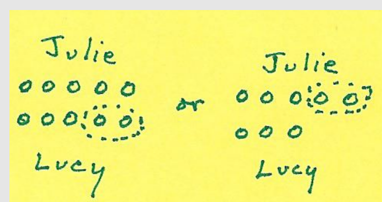
Representing and solving a new type of problem situation (Compare) In a Compare situation, two quantities are compared to find "How many more" or "How many less."•K.CC.6,K.CC.7 One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown.

• Compare problems build upon Kindergarten comparisons, in which students identified "Which is more?" or "Which is less?" without ascertaining the difference between the numbers.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

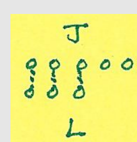
K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

Representing the difference in a Compare problem

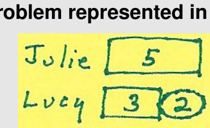


The language of comparisons is also difficult. For example, "Julie has three more apples than Lucy" tells both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form. Another language issue is that the comparing sentence might be stated in either of two related ways, using "more" or "less." Students need considerable experience with "less" to differentiate it from "more"; some children think that "less" means "more." Finally, as well as the basic "How many more/less" question form, the comparing sentence might take an active, equalizing and counterfactual form (e.g., "How many more apples does Lucy need to have as many as Julie?") or might be stated in a static and factual way as a question about how many things are unmatched (e.g., "If there are 8 trucks and 5 drivers, how many trucks do not have a driver?"). Extensive experience with a variety of contexts is needed to master these linguistic and situational complexities. Matching with objects and with drawings, and labeling each quantity (e.g., J or Julie and L or Lucy) is helpful. Later in Grade 1, a tape diagram can be used. These comparing diagrams can continue to be used for multi-digit numbers, fractions, decimals, and variables, thus connecting understandings of these numbers in

Compare problem solved by matching



Compare problem represented in tape diagram



comparing situations with such situations for single-digit numbers. The labels can get more detailed in later grades.

Some textbooks represent all Compare problems with a subtraction equation, but that is not how many students think of the subtypes. Students represent Compare situations in different ways, often as an unknown addend problem (see Table 1). If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context.

Representing and solving the subtypes for all unknowns in all three types In Grade 1, students solve problems of all twelve subtypes (see Table 2) including both language variants of Compare problems. Initially, the numbers in such problems are small enough that students can make math drawings showing all the objects in order to solve the problem. Students then represent problems with equations, called situation equations. For example, a situation equation for a Take From problem with Result Unknown might read $14 - 8 = \square$.

Put Together/Take Apart problems with Addend Unknown afford students the opportunity to see subtraction as the opposite of addition in a different way than as reversing the action, namely as finding an unknown addend.[1.OA.4] The meaning of subtraction as an unknown-addend addition problem is one of the essential understandings students will need in middle school in order to extend arithmetic to negative rational numbers.

[1.OA.4] Understand subtraction as an unknown-addend problem.

Students next gain experience with the more difficult and more "algebraic" problem subtypes in which a situation equation does not immediately lead to the answer. For example, a student analyzing a Take From problem with Change Unknown might write the situation equation $14 - \square = 8$. This equation does not immediately lead to the answer. To make progress, the student can write a related equation called a solution equation—in this case, either $8 + \square = 14$ or $14 - 8 = \square$. These equations both lead to the answer by Level 2 or Level 3 strategies (see discussion in the next section).

Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. Learning where the total is in addition equations (alone on one side of the equal sign) and in subtraction equations (to the left of the minus sign) helps students move from a situation equation to a related solution equation.

Because the language and conceptual demands are high, some students in Grade 1 may not master the most difficult subtypes of word problems, such as Compare problems that use language opposite to the operation required for solving (see the unshaded subtypes and variants in Table 2). Some students may also still have difficulty with the conceptual demands of Start Unknown problems. Grade 1 children should have an opportunity to solve and discuss such problems, but proficiency on grade level tests with these most difficult subtypes should wait until Grade 2 along with the other extensions of problem solving.

Using Level 2 and Level 3 strategies to extend addition and subtraction problem solving beyond 10, to problems within 20 As Grade 1 students are extending the range of problem types and subtypes they can solve, they are also extending the range of numbers they deal with[1.OA.6] and the sophistication of the methods they use to add and subtract within this larger range.[1.OA.1,1.OA.8]

The advance from Level 1 methods to Level 2 methods can be clearly seen in the context of situations with unknown addends.[1] These are the situations that can be represented by an addition equation with one unknown addend, e.g., $9 + \square = 13$. Students can solve some unknown addend problems by trial and error or by knowing the relevant decomposition of the total. But a Level 2 counting on solution involves seeing the 9 as part of 13, and understanding that counting the 9 things can be "taken as done" if we begin the count from 9: thus the student may say,

"Niiiiine,	ten,	eleven,	twelve,	thirteen."
	1	2	3	4

Students keep track of how many they counted on (here, 4) with fingers, mental images, or physical actions such as head bobs. Elongating the first counting word ("Niiiiine...") is natural and indicates that the student differentiates between the first addend and the counts for the second addend. Counting on enables students to add and subtract easily within 20 because they do not have to use fingers to show totals of more than 10 which is difficult. Students might also use the commutative property to shorten tasks, by counting on from the larger addend even if it is second (e.g., for $4 + 9$, counting on from 9 instead of from 4).

Counting on should be seen as a thinking strategy, not a rote method. It involves seeing the first addend as embedded in the total, and it involves a conceptual interplay between counting and the cardinality in the first addend (shifting from the cardinal meaning of the first addend to the counting meaning). Finally, there is a level of abstraction involved in counting on, because students are counting

[1] Grade 1 students also solve the easy Kindergarten problem subtypes by counting on.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

1.OA.8 Determine the unknown whole number in an addition or subtraction equation relating three whole numbers.

the words rather than objects. Number words have become objects to students.

Counting on can be used to add (find a total) or subtract (find an unknown addend). To an observer watching the student, adding and subtracting look the same. Whether the problem is $9 + 4$ or $13 - 9$, we will hear the student say the same thing: "Niiiiine, ten, eleven, twelve, thirteen" with four head bobs or four fingers unfolding. The differences are in what is being monitored to know when to stop, and what gives the answer.

Students in many countries learn counting forward methods of subtracting, including counting on. Counting on for subtraction is easier than counting down. Also, unlike counting down, counting on reinforces that subtraction is an unknown-addend problem. Learning to think of and solve subtractions as unknown addend problems makes subtraction as easy as addition (or even easier), and it emphasizes the relationship between addition and subtraction. The taking away meaning of subtraction can be emphasized within counting on by showing the total and then taking away the objects that are at the *beginning*. In a drawing this taking away can be shown with a horizontal line segment suggesting a minus sign. So one can think of the $9 + \square = 13$ situation as "I took away 9. I now have 10, 11, 12, 13 [stop when I hear 13], so 4 are left because I counted on 4 from 9 to get to 13." Taking away objects at the end suggests counting down, which is more difficult than counting on. Showing 13 decomposed in groups of five as in the illustration to the right also supports students seeing how to use the Level 3 make-a-ten method; 9 needs 1 more to make 10 and there are 3 more in 13, so 4 from 9 to 13.

Level 3 methods involve decomposing an addend and composing it with the other addend to form an equivalent but easier problem. This relies on properties of operations.[1.OA.3] Students do not necessarily have to justify their representations or solution using properties, but they can begin to learn to recognize these properties in action and discuss their use after solving.

There are a variety of methods to change to an easier problem. These draw on addition of three whole numbers.[1.OA.2] A known addition or subtraction can be used to solve a related addition or subtraction by decomposing one addend and composing it with the other addend. For example, a student can change $8+6$ to the easier $10 + 4$ by decomposing $6 = 2 + 4$ and composing the 2 with the 8 to make 10: $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$.

This method can also be used to subtract by finding an unknown addend: $14-8 = \square$, so $8+\square = 14$, so $14 = 8+2+4 = 8+6$, that is $14 - 8 = 6$. Students can think as for adding above (stopping when they reach 14), or they can think of taking 8 from 10, leaving 2 with the 4, which makes 6. One can also decompose with respect to ten: $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$, but this can be more difficult than the forward methods.

These make-a-ten methods• have three prerequisites reaching

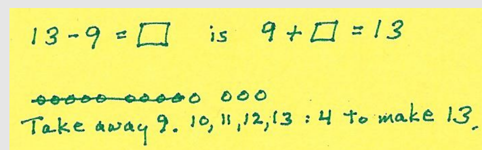
Counting on to add and subtract

$9 + 4$
"Niiiiine, ten, eleven, twelve, thirteen."
1 2 3 4

$13 - 9$
"Niiiiine, ten, eleven, twelve, thirteen."
1 2 3 4

When counting on to add $9 + 4$, the student is counting the fingers or head bobs to know when to stop counting aloud, and the last counting word said gives the answer. For counting on to subtract $13 - 9$, the opposite is true: the student is listening to counting words to know when to stop, and the accumulated fingers or head bobs give the answer.

"Taking away" indicated with horizontal line segment and solving by counting on to 13



1.OA.3 Apply properties of operations as strategies to add and subtract.

1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

• Computing 8 + 6 by making a ten

a. 8's partner to 10 is 2, so decompose 6 as 2 and its partner.

b. 2's partner to 6 is 4.

c. 10 + 4 is 14.

back to Kindergarten:

a. knowing the partner that makes 10 for any number (K.OA.4 sets the stage for this),

b. knowing all decompositions for any number below 10 (K.OA.3 sets the stage for this), and

c. knowing all teen numbers as $10 + n$ (e.g., $12 = 10 + 2$, $15 = 10 + 5$, see K.NBT.1 and 1.NBT.2b).

The make-a-ten methods are more difficult in English than in East Asian languages in which teen numbers are spoken as *ten, ten one, ten two, ten three*, etc. In particular, prerequisite c is harder in English because of the irregularities and reversals in the teen number words.•

Another Level 3 method that works for certain numbers is a doubles ± 1 or ± 2 method: $6+7 = 6+(6+1) = (6+6)+1 = 12+1 = 13$. These methods do not connect with place value the way make-a-ten methods do.

The Add To and Take From Start Unknown situations are particularly challenging with the larger numbers students encounter in Grade 1. The situation equation $\square + 6 = 15$ or $\square - 6 = 9$ can be rewritten to provide a solution. Students might use the commutative property of addition to change $\square + 6 = 15$ to $6 + \square = 15$, then count on or use Level 3 methods to compose 4 (to make ten) plus 5 (ones in the 15) to find 9. Students might reverse the action in the situation represented by $\square - 6 = 9$ so that it becomes $9 + 6 = \square$. Or they might use their knowledge that the total is the first number in a subtraction equation and the last number in an addition equation to rewrite the situation equation as a solution equation: $\square - 6 = 9$ becomes $9 + 6 = \square$ or $6 + 9 = \square$.

The difficulty levels in Compare problems differ from those in Put Together/Take Apart and Add To and Take From problems. Difficulties arise from the language issues mentioned before and especially from the opposite language variants where the comparing sentence suggests an operation opposite to that needed for the solution.•

As students progress to Level 2 and Level 3 methods, they no longer need representations that show each quantity as a group of objects. Students now move on to diagrams that use numbers and show relationships between these numbers. These can be extensions of drawings made earlier that did show each quantity as a group of objects. Add To/Take From situations at this point can continue to be represented by equations. Put Together/Take Apart situations can be represented by the example drawings shown in the margin. Compare situations can be represented with tape diagrams showing the compared quantities (one smaller and one larger) and the difference. Other diagrams showing two numbers and the unknown can also be used. Such diagrams are a major step forward because the same diagrams can represent the adding and subtracting situations

K.OA.4 For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4+1$).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

1.NBT.2b Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

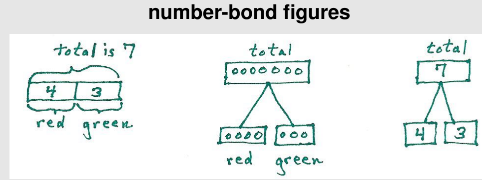
b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

• For example, "four" is spoken first in "fourteen," but this order is reversed in the numeral 14.

• Bigger Unknown: *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have?

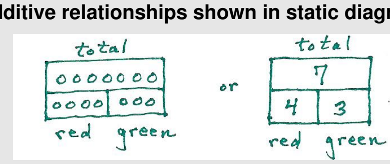
Smaller Unknown. *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have?

Additive relationship shown in tape, part-whole, and number-bond figures



The tape diagram shows the addends as the tapes and the total (indicated by a bracket) as a composition of those tapes. The part-whole diagram and number-bond diagram capture the composing-decomposing action to allow the representation of the total at the top and the addends at the bottom either as drawn quantities or as numbers.

Additive relationships shown in static diagrams



Students sometimes have trouble with static part-whole diagrams because these display a double representation of the total and the addends (the total 7 above and the addends 4 and 3 below), but at a given time in the addition or subtraction situation not all three quantities are present. The action of moving from the total to the addends (or from the addends to the total) in the number-bond diagram reduces this conceptual difficulty.

for all of the kinds of numbers students encounter in later grades (multi-digit whole numbers, fractions, decimals, variables). Students can also continue to represent any situation with a situation equation and connect such equations to diagrams.[MP1] Such connections can help students to solve the more difficult problem situation subtypes by understanding where the totals and addends are in the equation and rewriting the equation as needed.

[MP1] By relating equations and diagrams, students work toward this aspect of MP1: Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs.

Grade 2

Grade 2 students build upon their work in Grade 1 in two major ways.[2.OA.1] They represent and solve situational problems of all three types which involve addition and subtraction within 100 rather than within 20, and they represent and solve two-step situational problems of all three types.

2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Diagrams used in Grade 1 to show how quantities in the situation are related continue to be useful in Grade 2, and students continue to relate the diagrams to situation equations. Such relating helps students rewrite a situation equation like $\square - 38 = 49$ as $49 + 38 = \square$ because they see that the first number in the subtraction equation is the total. Each addition and subtraction equation has seven related equations. Students can write all of these equations, continuing to connect addition and subtraction, and their experience with equations of various forms.

Related addition and subtraction equations

$87 - 38 = 49$	$87 - 49 = 38$	$38 + 49 = 87$	$49 + 38 = 87$
$49 = 87 - 38$	$38 = 87 - 49$	$87 = 38 + 49$	$87 = 49 + 38$

Because there are so many problem situation subtypes, there are many possible ways to combine such subtypes to devise two-step problems. Because some Grade 2 students are still developing proficiency with the most difficult subtypes, two-step problems should not involve these subtypes. Most work with two-step problems should involve single-digit addends.

Most two-step problems made from two easy subtypes are easy to represent with an equation, as shown in the first two examples to the right. But problems involving a comparison or two middle difficulty subtypes may be difficult to represent with a single equation and may be better represented by successive drawings or some combination of a diagram for one step and an equation for the other (see the last three examples). Students can make up any kinds of two-step problems and share them for solving.

Examples of two-step Grade 2 word problems

Two easy subtypes with the same operation, resulting in problems represented as, for example, $9 + 5 + 7 = \square$ or $16 - 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 + 5 + 7$: There were 9 blue balls and 5 red balls in the bag. Aki put in 7 more balls. How many balls are in the bag altogether?

Two easy subtypes with opposite operations, resulting in problems represented as, for example, $9 - 5 + 7 = \square$ or $16 + 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 - 5 + 7$: There were 9 carrots on the plate. The girls ate 5 carrots. Mother put 7 more carrots on the plate. How many carrots are there now?

One easy and one middle difficulty subtype:

> For example: Maria has 9 apples. Corey has 4 fewer apples than Maria. How many apples do they have in all?

> For example: The zoo had 7 cows and some horses in the big pen. There were 15 animals in the big pen. Then 4 more horses ran into the big pen. How many horses are there now?

Two middle difficulty subtypes:

> For example: There were 9 boys and some girls in the park. In all, 15 children were in the park. Then some more girls came. Now there are 14 girls in the park. How many more girls came to the park?

The deep extended experiences students have with addition and subtraction in Kindergarten and Grade 1 culminate in Grade 2 with students becoming fluent in single-digit additions and the related subtractions using the mental Level 2 and 3 strategies as needed.[2.OA.2] So fluency in adding and subtracting single-digit numbers has progressed from numbers within 5 in Kindergarten to within 10 in Grade 1 to within 20 in Grade 2. The methods have also become more advanced.

2.OA.2 Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

The word *fluent* is used in the Standards to mean "fast and accurate." Fluency in each grade involves a mixture of just knowing some answers, knowing some answers from patterns (e.g., "adding 0 yields the same number"), and knowing some answers from the use of strategies. It is important to push sensitively and encouragingly toward fluency of the designated numbers at each grade level, recognizing that fluency will be a mixture of these kinds of thinking which may differ across students. The extensive work relating addition and subtraction means that subtraction can frequently be solved by thinking of the related addition, especially for smaller numbers. It is also important that these patterns, strategies and decompositions still be available in Grade 3 for use in multiplying and dividing and in distinguishing adding and subtracting from multiplying and dividing. So the important press toward fluency should also allow students to fall back on earlier strategies when needed. By the end of the K–2 grade span, students have sufficient experience with addition and subtraction to know single-digit sums from memory;[2.OA.2] as should be clear from the foregoing, this is not a matter of instilling facts divorced from their meanings, but rather as an outcome of a multi-year process that heavily involves the interplay of practice and reasoning.

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

Extensions to other standard domains and to higher grades In Grades 2 and 3, students continue and extend their work with adding and subtracting situations to length situations[2.MD.5,2.MD.6] (addition and subtraction of lengths is part of the transition from whole number addition and subtraction to fraction addition and subtraction) and to bar graphs.[2.MD.10,3.MD.3] Students solve two-step[3.OA.8] and multistep[4.OA.3] problems involving all four operations. In Grades 3, 4, and 5, students extend their understandings of addition and subtraction problem types in Table 1 to situations that involve fractions and decimals. Importantly, the situational meanings for addition and subtraction remain the same for fractions and decimals as for whole numbers.

[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.10] Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

[3.MD.3] Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.8] Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

[4.OA.3] Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Summary of K–2 Operations and Algebraic Thinking

Kindergarten Students in Kindergarten work with three kinds of problem situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown. The numbers in these problems involve addition and subtraction within 10. Students represent these problems with concrete objects and drawings, and they find the answers by counting (Level 1 method). More specifically,

- For Add To with Result Unknown, they make or draw the starting set of objects and the change set of objects, and then they count the total set of objects to give the answer.
- For Take From with Result Unknown, they make or draw the starting set and "take away" the change set; then they count the remaining objects to give the answer.
- For Put Together/Take Apart with Total Unknown, they make or draw the two addend sets, and then they count the total number of objects to give the answer.

Grade 1 Students in Grade 1 work with all of the problem situations, including all subtypes and language variants. The numbers in these problems involve additions involving single-digit addends, and the related subtractions. Students represent these problems with math drawings and with equations.

Students master the majority of the problem types. They might sometimes use trial and error to find the answer, or they might just know the answer based on previous experience with the given numbers. But as a general method they learn how to find answers to these problems by counting on (a Level 2 method), and they understand and use this method.[1.OA.5,1.OA.6] Students also work with Level 3 methods that change a problem to an easier equivalent problem.[1.OA.3,1.OA.6] The most important of these Level 3 methods involve making a ten, because these methods connect with the place value concepts students are learning in this grade (see the NBT Progression) and work for any numbers. Students also solve the easier problem subtypes with these Level 3 methods.

The four problem subtypes that Grade 1 students should work with, but need not master, are:

- Add To with Start Unknown
- Take From with Start Unknown
- Compare with Bigger Unknown using "fewer" language (misleading language suggesting the wrong operation)
- Compare with Smaller Unknown using "more" language (misleading language suggesting the wrong operation)

1.OA.5 Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

1.OA.3 Apply properties of operations as strategies to add and subtract.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

Grade 2 Students in Grade 2 master all of the problem situations and all of their subtypes and language variants. The numbers in these problems involve addition and subtraction within 100. They represent these problems with diagrams and/or equations. For problems involving addition and subtraction within 20, more students master Level 3 methods; increasingly for addition problems, students might just know the answer (by end of Grade 2, students know all sums of two-digit numbers from memory[2.OA.2]). For other problems involving numbers to 100, Grade 2 students use their developing place value skills and understandings to find the answer (see the NBT Progression). Students work with two-step problems, especially with single-digit addends, but do not work with two-step problems in which both steps involve the most difficult problem subtypes and variants.

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

Grade 3

Students focus on understanding the meaning and properties of multiplication and division and on finding products of single-digit multiplying and related quotients.[3.OA.1–7] These skills and understandings are crucial; students will rely on them for years to come as they learn to multiply and divide with multi-digit whole number and to add, subtract, multiply and divide with fractions and with decimals. Note that mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding,[3.OA.7] may be quite time consuming because there are no general strategies for multiplying or dividing all single-digit numbers as there are for addition and subtraction. Instead, there are many patterns and strategies dependent upon specific numbers. So it is imperative that extra time and support be provided if needed.

Common types of multiplication and division situations. Common multiplication and division situations are shown in Table 3. There are three major types, shown as rows of Table 3. The Grade 3 standards focus on Equal Groups and on Arrays.• As with addition and subtraction, each multiplication or division situation involves three quantities, each of which can be the unknown. Because there are two factors and one product in each situation (product = factor × factor), each type has one subtype solved by multiplication (Unknown Product) and two unknown factor subtypes solved by division.

3.OA.1 Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

3.OA.2 Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.

3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

3.OA.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers.

3.OA.5 Apply properties of operations as strategies to multiply and divide.

3.OA.6 Understand division as an unknown-factor problem.

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

• Multiplicative Compare situations are more complex than Equal Groups and Arrays, and must be carefully distinguished from additive Compare problems. Multiplicative comparison first enters the Standards at Grade 4.[4.OA.1] For more information on multiplicative Compare problems, see the Grade 4 section of this progression.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

Table 3: Multiplication and division situations

	$A \times B = \square$	$A \times \square = C$ and $C \div A = \square$	$\square \times B = C$ and $C \div B = \square$
Equal Groups of Objects	Unknown Product There are A bags with B plums in each bag. How many plums are there in all?	Group Size Unknown If C plums are shared equally into A bags, then how many plums will be in each bag?	Number of Groups Unknown If C plums are to be packed B to a bag, then how many bags are needed?
Arrays of Objects		*Equal groups language*	
	Unknown Product There are A rows of apples with B apples in each row. How many apples are there?	Unknown Factor If C apples are arranged into A equal rows, how many apples will be in each row?	Unknown Factor If C apples are arranged into equal rows of B apples, how many rows will there be?
		Row and column language	
	Unknown Product The apples in the grocery window are in A rows and B columns. How many apples are there?	Unknown Factor If C apples are arranged into an array with A rows, how many columns of apples are there?	Unknown Factor If C apples are arranged into an array with B columns, how many rows are there?
Compare		$A > 1$	
	Larger Unknown A blue hat costs \$$B$. A red hat costs A times as much as the blue hat. How much does the red hat cost?	Smaller Unknown A red hat costs \$$C$ and that is A times as much as a blue hat costs. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. How many times as much does the red hat cost as the blue hat?
		$A < 1$	
	Smaller Unknown A blue hat costs \$$B$. A red hat costs A as much as the blue hat. How much does the red hat cost?	Larger Unknown A red hat costs \$$C$ and that is A of the cost of a blue hat. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. What fraction of the cost of the blue hat is the cost of the red hat?

Adapted from box 2–4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

Notes

Equal groups problems can also be stated in terms of columns, exchanging the order of A and B, so that the same array is described. For example: There are B columns of apples with A apples in each column. How many apples are there?

In the row and column situations (as with their area analogues), number of groups and group size are not distinguished.

Multiplicative Compare problems appear first in Grade 4, with whole-number values for A, B, and C, and with the "times as much" language in the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit fraction language change the subject of the comparing sentence, e.g., "A red hat costs A times as much as the blue hat" results in the same comparison as "A blue hat costs $1/A$ times as much as the red hat," but has a different subject.

In Equal Groups, the roles of the factors differ. One factor is the number of objects in a group (like any quantity in addition and subtraction situations), and the other is a multiplier that indicates the number of groups. So, for example, 4 groups of 3 objects is arranged differently than 3 groups of 4 objects. Thus there are two kinds of division situations depending on which factor is the unknown (the number of objects in each group or the number of groups). In the Array situations, the roles of the factors do not differ. One factor tells the number of rows in the array, and the other factor tells the number of columns in the situation. But rows and columns depend on the orientation of the array. If an array is rotated 90°, the rows become columns and the columns become rows. This is useful for seeing the commutative property for multiplication[3.OA.5] in rectangular arrays and areas. This property can be seen to extend to Equal Group situations when Equal Group situations are related to arrays by arranging each group in a row and putting the groups under each other to form an array. Array situations can be seen as Equal Group situations if each row or column is considered as a group. Relating Equal Group situations to Arrays, and indicating rows or columns within arrays, can help students see that a corner object in an array (or a corner square in an area model) is not double counted: at a given time, it is counted as part of a row or as a part of a column but not both.

As noted in Table 3, row and column language can be difficult. The Array problems given in the table are of the simplest form in which a row is a group and Equal Groups language is used ("with 6 apples in each row"). Such problems are a good transition between the Equal Groups and array situations and can support the generalization of the commutative property discussed above. Problems in terms of "rows" and "columns," e.g., "The apples in the grocery window are in 3 rows and 6 columns," are difficult because of the distinction between the number of things *in a* row and the number *of* rows. There are 3 rows but the number of columns (6) tells how many are in each row. There are 6 columns but the number of rows (3) tells how many are in each column. Students do need to be able to use and understand these words, but this understanding can grow over time while students also learn and use the language in the other multiplication and division situations.

Variations of each type that use measurements instead of discrete objects are given in the Measurement and Data Progression. Grade 2 standards focus on length measurement[2.MD.1–4] and Grade 3 standards focus on area measurement.[3.MD.5–7] The measurement examples are more difficult than are the examples about discrete objects, so these should follow problems about discrete objects. Area problems where regions are partitioned by unit squares are foundational for Grade 3 standards because area is used as a model for single-digit multiplication and division strategies,[3.MD.7] in Grade 4 as a model for multi-digit multiplication and division and in Grade 5 and Grade 6 as a model for multiplication and division of decimals

[3.OA.5] Apply properties of operations as strategies to multiply and divide.

[2.MD.1] Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

[2.MD.2] Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

[2.MD.3] Estimate lengths using units of inches, feet, centimeters, and meters.

[2.MD.4] Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

[3.MD.5] Recognize area as an attribute of plane figures and understand concepts of area measurement.

[3.MD.6] Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

[3.MD.7] Relate area to the operations of multiplication and addition.

and of fractions.[5.NBT.6] The distributive property is central to all of these uses and will be discussed later.

The top row of Table 3 shows the usual order of writing multiplications of Equal Groups in the United States. The equation $3 \times 6 = \square$ means how many are in 3 groups of 6 things each: three sixes. But in many other countries the equation $3 \times 6 = \square$ means how many are 3 things taken 6 times (6 groups of 3 things each): six threes. Some students bring this interpretation of multiplication equations into the classroom. So it is useful to discuss the different interpretations and allow students to use whichever is used in their home. This is a kind of linguistic commutativity that precedes the reasoning discussed above arising from rotating an array. These two sources of commutativity can be related when the rotation discussion occurs.

5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Levels in problem representation and solution Multiplication and division problem representations and solution methods can be considered as falling within three levels related to the levels for addition and subtraction (see Appendix). Level 1 is making and counting all of the quantities involved in a multiplication or division. As before, the quantities can be represented by objects or with a diagram, but a diagram affords reflection and sharing when it is drawn on the board and explained by a student. The Grade 2 standards 2.OA.3 and 2.OA.4 are at this level but set the stage for Level 2. Standard 2.OA.3 relates doubles additions up to 20 to the concept of odd and even numbers and to counting by 2s (the easiest count-by in Level 2) by pairing and counting by 2s the things in each addend. 2.OA.4 focuses on using addition to find the total number of objects arranged in rectangular arrays (up to 5 by 5).

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Level 2 is repeated counting on by a given number, such as for 3: $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.

Supporting Level 2 methods with arrays

Small arrays (up to 5×5) support seeing and beginning to learn the Level 2 count-bys for the first five equal groups of the small numbers 2 through 5 if the running total is written to the right of each row (e.g., $3, 6, 9, 12, 15$). Students may write repeated additions and then count by ones without the objects, often emphasizing each last number said for each group. Grade 3 students can be encouraged to move as early as possible from equal grouping or array models that show all of the quantities to similar representations using diagrams that show relationships of numbers because diagrams are faster and less error-prone and support methods at Level 2 and Level 3. Some demonstrations of methods or of properties may need to fall back to initially showing all quantities along with a diagram.

The difficulty of saying and remembering the count-by for a given number depends on how closely related it is to 10, the base for our written and spoken numbers. For example, the count-by sequence for 5 is easy, but the count-by sequence for 7 is difficult. Decomposing with respect to a ten can be useful in going over a decade within a count-by. For example in the count-by for 7, students might use the following mental decompositions of 7 to compose up to and then go over the next decade, e.g., $14 + 7 = 14 + 6 + 1 = 20 + 1 = 21$. The count-by sequence can also be said with the factors, such as "one times three is *three*, two times three is *six*, three times three is *nine*, etc." Seeing as well as hearing the count-bys and the equations for

Composing up to, then over the next decade

7	14	21	28	35	42	49	56	63	70
	$6+1$		$2+5$	$5+2$		$1+6$	$4+3$		

There is an initial $3 + 4$ for $7 + 7$ that completes the reversing pattern of the partners of 7 involved in these mental decompositions with respect to the decades.

the multiplications or divisions can be helpful.

Level 3 methods use the associative property or the distributive property to compose and decompose. These compositions and decompositions may be additive (as for addition and subtraction) or multiplicative. For example, students multiplicatively compose or decompose:

> 4×6 is easier to count by 3 eight times:
>
> $$4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$$

Students may know a product 1 or 2 ahead of or behind a given product and say:

> I know 6×5 is 30, so 7×5 is $30 + 5$ more which is 35.
> This implicitly uses the distributive property:
>
> $$7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 3 = 30 + 5 = 35.$$

Students may decompose a product that they do not know in terms of two products they know (for example, 4×7 shown in the margin).

Students may not use the properties explicitly (for example, they might omit the second two steps), but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

The $5 + n$ pattern students used earlier for additions can now be extended to show how 6, 7, 8, and 9 times a number are $5 + 1$, $5 + 2$, $5 + 3$, and $5 + 4$ times that number. These patterns are particularly easy to do mentally for the numbers 4, 6, and 8. The 9s have particularly rich patterns based on $9 = 10 - 1$. The pattern of the tens digit in the product being 1 less than the multiplier, the ones digit in the product being 10 minus the multiplier, and that the digits in nines products sum to 9 all come from this pattern.

There are many opportunities to describe and reason about the many patterns involved in the Level 2 count-bys and in the Level 3 composing and decomposing methods. There are also patterns in multiplying by 0 and by 1. These need to be differentiated from the patterns for adding 0 and adding 1 because students often confuse these three patterns: $n + 0 = n$ but $n \times 0 = 0$, and $n \times 1$ is the pattern that does not change n (because $n \times 1 = n$). Patterns make multiplication by some numbers easier to learn than multiplication by others, so approaches may teach multiplications and divisions in various orders depending on what numbers are seen as or are supported to be easiest.

Multiplications and divisions can be learned at the same time and can reinforce each other. Level 2 methods can be particularly easy for division, as discussed above. Level 3 methods may be more difficult for division than for multiplication.

Throughout multiplication and division learning, students gain fluency and begin to know certain products and unknown factors.

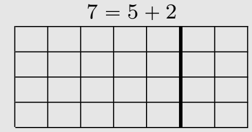
Decomposing 4×7

$$\begin{aligned} 4 \times 7 &= 4 \times (5 + 2) \\ &= (4 \times 5) + (4 \times 2) \\ &= 20 + 8 \\ &= 28 \end{aligned}$$

Supporting reasoning with area diagram

$7 = 5 + 2$

4



$20 + 8 = 28$
$4 \times 5 + 4 \times 2 = 4 \times 7$

The $5 + n$ pattern for multiplying the numbers 4, 6, and 8

n		$4 \times n$		$6 \times n$		$8 \times n$	
1	$5+1$	4	24	6	36	8	48
2	$5+2$	8	28	12	42	16	56
3	$5+3$	12	32	18	48	24	64
4	$5+4$	16	36	24	54	32	72
5	$5+5$	20	40	30	60	40	80

Patterns in multiples of 9

$1 \times 9 = 9$

$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$

$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$ etc

All of the understandings of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single-digit numbers and 10.[3.OA.7] Such fluency may be reached by becoming fluent for each number (e.g., the 2s, the 5s, etc.) and then extending the fluency to several, then all numbers mixed together. Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. As should be clear from the foregoing, this isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning. All of the work on how different numbers fit with the base-ten numbers culminates in these "just know" products and is necessary for learning products. Fluent dividing for all single-digit numbers, which will combine just knows, knowing from a multiplication, patterns, and best strategy, is also part of this vital standard.

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Using a letter for the unknown quantity, the order of operations, and two-step word problems with all four operations Students in Grade 3 begin the step to formal algebraic language by using a letter for the unknown quantity in expressions or equations for one- and two-step problems.[3.OA.8] But the symbols of arithmetic, $\times$ or $\cdot$ or $*$ for multiplication and $\div$ or / for division, continue to be used in Grades 3, 4, and 5.

3.OA.8 Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Understanding and using the associative and distributive properties (as discussed above) requires students to know two conventions for reading an expression that has more than one operation:

1. Do the operation inside the parentheses before an operation outside the parentheses (the parentheses can be thought of as hands curved around the symbols and grouping them).

2. If a multiplication or division is written next to an addition or subtraction, imagine parentheses around the multiplication or division (it is done before these operations). At Grades 3 through 5, parentheses can usually be used for such cases so that fluency with this rule can wait until Grade 6.

MP7 Making use of structure to make computation easier:

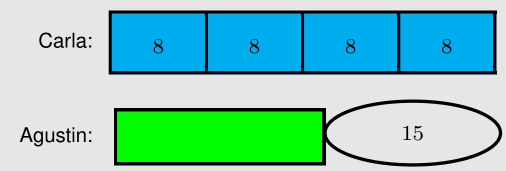
$$13 + 29 + 77 + 11 = (13 + 77) + (29 + 11)$$

These conventions are often called the Order of Operations and can seem to be a central aspect of algebra. But actually they are just simple "rules of the road" that allow expressions involving more than one operation to be interpreted unambiguously and thus are connected with the mathematical practice of communicating precisely.[MP6] Use of parentheses is important in displaying structure and thus is connected with the mathematical practice of making use of structure.[MP7] Parentheses are important in expressing the associative and especially the distributive properties. These properties are at the heart of Grades 3 to 5 because they are used in the Level 3 multiplication and division strategies, in multi-digit and decimal multiplication and division, and in all operations with fractions.

As with two-step problems at Grade 2,[2.OA.1, 2.MD.5] which involve only addition and subtraction, the Grade 3 two-step word problems vary greatly in difficulty and ease of representation. More difficult problems may require two steps of representation and solution rather than one. Use of two-step problems involving easy or middle difficulty adding and subtracting within 1,000 or one such adding or subtracting with one step of multiplication or division can help to maintain fluency with addition and subtraction while giving the needed time to the major Grade 3 multiplication and division standards.

A two-step problem with diagram showing problem situation and equations showing the two parts

Carla has 4 packages of silly bands. Each package has 8 silly bands in it. Agustin is supposed to get 15 fewer silly bands than Carla. How many silly bands should Agustin get?



$C =$ number of Carla's silly bands
$A =$ number of Agustin's silly bands

$$C = 4 \times 8 = 32$$

$$\begin{aligned} A + 15 &= C \\ A + 15 &= 32 \\ A &= 17 \end{aligned}$$

Students may be able to solve this problem without writing such equations.

Grade 4

Multiplication Compare Consider two diving boards, one 40 feet high, the other 8 feet high. Students in earlier grades learned to compare these heights in an additive sense—"This one is 32 feet higher than that one"—by solving additive Compare problems[2.OA.1] and using addition and subtraction to solve word problems involving length.[2.MD.5] Students in Grade 4 learn to compare these quantities multiplicatively as well: "This one is 5 times as high as that one."[4.OA.1,4.OA.2,4.MD.1,4.MD.2] In an additive comparison, the underlying question is *what amount would be added to one quantity* in order to result in the other. In a multiplicative comparison, the underlying question is *what factor would multiply one quantity* in order to result in the other. Multiplication Compare situations are shown in Table 3.

Language can be difficult in Multiplication Compare problems. The language used in the three examples in Table 3 is fairly simple, e.g., "A red hat costs 3 times as much as the blue hat." Saying the comparing sentence in the opposite way is more difficult. It could be said using division, e.g., "The cost of a red hat divided by 3 is the cost of a blue hat." It could also be said using a unit fraction, e.g., "A blue hat costs one-third as much as a red hat"; note however that multiplying by a fraction in not an expectation of the Standards in Grade 4. In any case, many languages do not use either of these options for saying the opposite comparison. They use the terms *three times more than* and *three times less than* to describe opposite multiplicative comparisons. These did not used to be acceptable usages in English because they mix the multiplicative and additive comparisons and are ambiguous. If the cost of a red hat is three times more than a blue hat that costs $5, does a red hat cost $15 (three times as much) or $20 (three times more than: a difference that is three times as much)? However, the terms *three times more than* and *three times less than* are now appearing frequently in newspapers and other written materials. It is recommended to discuss these complexities with Grade 4 students while confining problems that appear on tests or in multi-step problems to the well-defined multiplication language in Table 3. The tape diagram for the additive Compare situation that shows a smaller and a larger tape can be extended to the multiplication Compare situation.

Fourth graders extend problem solving to multi-step word problems using the four operations posed with whole numbers. The same limitations discussed for two-step problems concerning representing such problems using equations apply here. Some problems might easily be represented with a single equation, and others will be more sensibly represented by more than one equation or a diagram and one or more equations. Numbers can be those in the Grade 4 standards, but the number of steps should be no more than three and involve only easy and medium difficulty addition and subtraction problems.

2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

2.MD.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

4.OA.2 Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

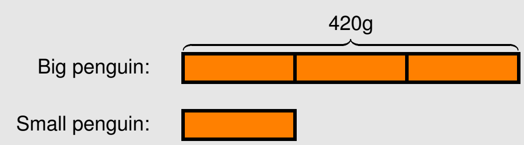
Tape diagram used to solve the Compare problem in Table 3

B is the cost of a blue hat in dollars
R is the cost of a red hat in dollars

$\boxed{\$6}$	$3 \times B = R$
$\boxed{\$6}\boxed{\$6}\boxed{\$6}$	$3 \times \$6 = \$18$

A tape diagram used to solve a Compare problem

A big penguin will eat 3 times as much fish as a small penguin. The big penguin will eat 420 grams of fish. All together, how much will the two penguins eat?



B = number of grams the big penguin eats
S = number of grams the small penguin eats

$$3 \cdot S = B$$
$$3 \cdot S = 420$$
$$S = 140$$

$$S + B = 140 + 420$$
$$= 560$$

Remainders In problem situations, students must interpret and use remainders with respect to context.[4.OA.3] For example, what is the smallest number of busses that can carry 250 students, if each bus holds 36 students? The whole number quotient in this case is 6 and the remainder is 34; the equation $250 = 6 \times 36 + 34$ expresses this result and corresponds to a picture in which 6 busses are completely filled while a seventh bus carries 34 students. Notice that the answer to the stated question (7) differs from the whole number quotient.

On the other hand, suppose 250 pencils were distributed among 36 students, with each student receiving the same number of pencils. What is the largest number of pencils each student could have received? In this case, the answer to the stated question (6) is the same as the whole number quotient. If the problem had said that the teacher got the remaining pencils and asked how many pencils the teacher got, then the remainder would have been the answer to the problem.

[4.OA.3] Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Factors, multiples, and prime and composite numbers Students extend the idea of decomposition to multiplication and learn to use the term *multiple*.[4.OA.4] Any whole number is a multiple of each of its factors, so for example, 21 is a multiple of 3 and a multiple of 7 because $21 = 3 \cdot 7$. A number can be multiplicatively decomposed into equal groups and expressed as a product of these two factors (called factor pairs). A prime number has only one and itself as factors. A composite number has two or more factor pairs. Students examine various patterns in factor pairs by finding factor pairs for all numbers 1 to 100 (e.g., no even number other than 2 will be prime because it always will have a factor pair including 2). To find all factor pairs for a given number, students can search systematically, by checking if 2 is a factor, then 3, then 4, and so on, until they start to see a "reversal" in the pairs (for example, after finding the pair 6 and 9 for 54, students will next find the reverse pair, 9 and 6; all subsequent pairs will be reverses of previously found pairs). Students understand and use of the concepts and language in this area, but need not be fluent in finding all factor pairs. Determining whether a given whole number in the range 1 to 100 is a multiple of a given one-digit number is a matter of interpreting prior knowledge of division in terms of the language of multiples and factors.

[4.OA.4] Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Generating and analyzing patterns This standard[4.OA.5] begins a small focus on reasoning about number or shape patterns, connecting a rule for a given pattern with its sequence of numbers or shapes. Patterns that consist of repeated sequences of shapes or growing sequences of designs can be appropriate for the grade. For example, students could examine a sequence of dot designs in which each design has 4 more dots than the previous one and they could reason about how the dots are organized in the design to determine the

[4.OA.5] Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

total number of dots in the 100th design. In examining numerical sequences, fourth graders can explore rules of repeatedly adding the same whole number or repeatedly multiplying by the same whole number. Properties of repeating patterns of shapes can be explored with division. For example, to determine the 100th shape in a pattern that consists of repetitions of the sequence “square, circle, triangle,” the fact that when we divide 100 by 3 the whole number quotient is 33 with remainder 1 tells us that after 33 full repeats, the 99th shape will be a triangle (the last shape in the repeating pattern), so the 100th shape is the first shape in the pattern, which is a square. Notice that the Standards do not require students to infer or guess the underlying rule for a pattern, but rather ask them to generate a pattern from a given rule and identify features of the given pattern.

Grade 5

As preparation for the Expressions and Equations Progression in the middle grades, students in Grade 5 begin working more formally with expressions.[5.OA.1, 5.OA.2] They write expressions to express a calculation, e.g., writing $2 \times (8+7)$ to express the calculation "add 8 and 7, then multiply by 2." They also evaluate and interpret expressions, e.g., using their conceptual understanding of multiplication to interpret $3 \times (18932 + 921)$ as being three times as large as $18932 + 921$, without having to calculate the indicated sum or product. Thus, students in Grade 5 begin to think about numerical expressions in ways that prefigure their later work with variable expressions (e.g., three times an unknown length is $3 \cdot L$). In Grade 5, this work should be viewed as exploratory rather than for attaining mastery; for example, expressions should not contain nested grouping symbols, and they should be no more complex than the expressions one finds in an application of the associative or distributive property, e.g., $(8+27)+2$ or $(6 \times 30) + (6 \times 7)$. Note however that the numbers in expressions need not always be whole numbers.

Students extend their Grade 4 pattern work by working briefly with two numerical patterns that can be related and examining these relationships within sequences of ordered pairs and in the graphs in the first quadrant of the coordinate plane.[5.OA.3] This work prepares students for studying proportional relationships and functions in middle school.

[5.OA.1] Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

[5.OA.2] Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

[5.OA.3] Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Connections to NF and NBT in Grades 3 through 5

Students extend their whole number work with adding and subtracting and multiplying and dividing situations to decimal numbers and fractions. Each of these extensions can begin with problems that include all of the subtypes of the situations in Tables 1 and 2. The operations of addition, subtraction, multiplication, and division continue to be used in the same way in these problem situations when they are extended to fractions and decimals (although making these extensions is not automatic or easy for all students). The connections described for Kindergarten through Grade 3 among word problem situations, representations for these problems, and use of properties in solution methods are equally relevant for these new kinds of numbers. Students use the new kinds of numbers, fractions and decimals, in geometric measurement and data problems and extend to some two-step and multi-step problems involving all four operations. In order to keep the difficulty level from becoming extreme, there should be a tradeoff between the algebraic or situational complexity of any given problem and its computational difficulty taking into account the kinds of numbers involved.

As students' notions of quantity evolve and generalize from discrete to continuous during Grades 3–5, their notions of multiplication evolves and generalizes. This evolution deserves special attention because it begins in OA but ends in NF. Thus, the concept of multiplication begins in Grade 3 with an entirely discrete notion of "equal groups."[3.OA.1] By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion "times as much."[4.OA.1] This notion has more affinity to continuous quantities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as $\frac{3}{4}$ cup of flour.[4.NF.4,4.MD.2] By Grade 5, when students multiply fractions in general,[5.NF.4] products can be larger or smaller than either factor, and multiplication can be seen as an operation that "stretches or shrinks" by a scale factor.[5.NF.5] This view of multiplication as scaling is the appropriate notion for reasoning multiplicatively with continuous quantities.

[3.OA.1] Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

Where the Operations and Algebraic Thinking Progression is heading

Connection to the Number System The properties of and relationships between operations that students worked with in Grades K–5 will become even more prominent in extending arithmetic to systems that include negative numbers; meanwhile the meanings of the operations will continue to evolve, e.g., subtraction will become "adding the opposite."

Connection to Expressions and Equations In Grade 6, students will begin to view expressions not just as calculation recipes but as entities in their own right, which can be described in terms of their parts. For example, students see $8 * (5 + 2)$ as the product of 8 with the sum $5 + 2$. In particular, students must use the conventions for order of operations to *interpret* expressions, not just to evaluate them. Viewing expressions as entities created from component parts is essential for seeing the structure of expressions in later grades and using structure to reason about expressions and functions.

As noted above, the foundation for these later competencies is laid in Grade 5 when students write expressions to record a "calculation recipe" without actually evaluating the expression, use parentheses to formulate expressions, and examine patterns and relationships numerically and visually on a coordinate plane graph.[5.OA.1,5.OA.2] Before Grade 5, student thinking that also builds toward the Grade 6 EE work is focusing on the expressions on each side of an equation, relating each expression to the situation, and discussing the situational and mathematical vocabulary involved to deepen the understandings of expressions and equations.

[5.OA.1] Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

[5.OA.2] Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

In Grades 6 and 7, students begin to explore the systematic algebraic methods used for solving algebraic equations. Central to these methods are the relationships between addition and subtraction and between multiplication and division, emphasized in several parts of this Progression and prominent also in the 6–8 Progression for the Number System. Students' varied work throughout elementary school with equations with unknowns in all locations and in writing equations to decompose a given number into many pairs of addends or many pairs of factors are also important foundations for understanding equations and for solving equations with algebraic methods. Of course, any method of solving, whether systematic or not, relies on an understanding of what solving itself is—namely, a process of answering a question: which values from a specified set, if any, make the equation true?[6.EE.5]

[6.EE.5] Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Students represent and solve word problems with equations involving one unknown quantity in K through 5. The quantity was expressed by a □ or other symbol in K–2 and by a letter in Grades 3 to 5. Grade 6 students continue the K–5 focus on representing a problem situation using an equation (a situation equation) and then (for the more difficult situations) writing an equivalent equation that is easier to solve (a solution equation). Grade 6 students discuss their reasoning more explicitly by focusing on the structures of expressions and using the properties of operations explicitly. Some of the math drawings that students have used in K through 5 to represent problem situations continue to be used in the middle grades. These can help students throughout the grades deepen the connections they make among the situation and problem representations by a drawing and/or by an equation, and support the informal K–5 and increasingly formal 6–8 solution methods arising from understanding the structure of expressions and equations.

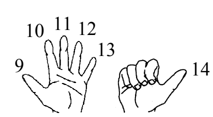
Appendix. Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away.

Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Adding ($8 + 6 = \square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown.

Subtracting ($14 - 8 = \square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown added. Use this strategy for Take From/Result Unknown.

Levels	**8 + 6 = 14**	**14 – 8 = 6**
Level 1: Count all	Count All a: 1 2 3 4 5 6 7 8 / b: 1 2 3 4 5 6 ○○○○○ ○○○ ○○○○○ ○ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~○○ ○○○○ b: 1 2 3 4 5 6 7 8 / c: 1 2 3 4 5 6
Level 2: Count on	Count On 8 ○○○○○ ○○○ ○○○○○ ○ 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten ○○○○○ ○○○ ○○ ○○○ ○ 10 + 4 ○○○○○ ○○○ ○○○○○ ○ 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~○○○○○ ○○○~~○○ ○○○○ 8 + 2 + 4 6 8 + 6 = 14
Doubles ± *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2. Counting On.

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on can be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words and extended experience with Level 1 methods in Kindergarten.

> Adding (e. g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
>
> Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The keeping track method is monitored so that counting on stops when the known total has been reached. The keeping track method tells the unknown addend.
>
> Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$ and uses counting on to find an unknown addend (as above).

The problems in Table 2 which are solved by Level 1 methods in Kindergarten can also be solved using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting).

The middle difficulty (lightly shaded) problem types in Table 2 for Grade 1 are directly accessible with the embedded thinking of Level 2 methods and can be solved by counting on.

> Finding an unknown addend (e.g., $8 + \square = 14$) is used for Add To/Change Unknown, Put Together/Take Apart/Addend Unknown, and Compare/Difference Unknown. It is also used for Take From/Change Unknown ($14 - \square =$

8) after a student has decomposed the total into two addends, which means they can represent the situation as $14 - 8 = \square$.

Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in Table 2). Grade 1 students do not necessarily master the Compare Bigger Unknown or Smaller Unknown problems with the misleading language in the bottom row of Table 2.

Solving an equation such as $6 + 8 = \square$ by counting on from 8 relies on the understanding that $8 + 6$ gives the same total, an implicit use of the commutative property without the accompanying written representation $6 + 8 = 8 + 6$.

Level 3. Convert to an Easier Equivalent Problem.

Decompose an addend and compose a part with another addend.

These methods can be used to add or to find an unknown addend (and thus to subtract). These methods implicitly use the associative property.

Adding

Make a ten. E.g, for $8 + 6 = \square$,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so $8 + 6$ becomes $10 + 4$.

Doubles plus or minus 1. E.g., for $6 + 7 = \square$,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so $6 + 7$ becomes $12 + 1$.

Finding an unknown addend

Make a ten. E. g., for $8 + \square = 14$,

$$8 + \underline{2} = 10 \text{ and } \underline{4} \text{ more makes } 14.\ \underline{2 + 4} = 6.$$

So $8 + \square = 14$ is done as two steps: how many up to ten and how many over ten (which can be seen in the ones place of 14).

Doubles plus or minus 1. E.g., for $6 + \square = 13$,

$$6 + \underline{6 + 1} = 12 + 1.\ \underline{6 + 1} = 7.$$

So $6 + \square = 13$ is done as two steps: how many up to 12 $(6 + 6)$ and how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend. E.g., solve $14-8=\square$ or $13-6=\square$ as $8+\square=14$ or $6+\square=13$ by the above methods (make a ten or doubles plus or minus 1).

Make a ten by going down over ten. E.g., $14-8=\square$ can be done in two steps by going down over ten: $14-4$ (to get to 10) $-4=6$.

The Level 1 and Level 2 problem types can be solved using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend as shown above by methods at any level, but usually at Level 2 or 3. Many students only show in their writing part of this multi-step process of re-representing the situation.

> Students re-represent Add To/Start Unknown $\square+6=14$ situations as $6+\square=14$ by using the commutative property (formally or informally).
>
> Students re-represent Take From/Start Unknown $\square-8=6$ situations by reversing as $6+8=\square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare misleading language situations can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct solution by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same diagrams used for the other versions of compare situations; focusing on which quantity is bigger and which is smaller helps to overcome the misleading language.

Some students may solve Level 3 problem types by doing the above re-representing but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, facilitating such re-representing. Labels on diagrams can help connect the parts of the diagram to the corresponding parts of the situation. But students may know and understand things that they may not use for a given solution of a problem as they increasingly do various representing and re-representing steps mentally.

K–5, Number and Operations in Base Ten

Progressions for the Common Core State Standards in Mathematics (draft)

21 April 2012

K–5, Number and Operations in Base Ten

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding in computations, decomposing 1 ten into 10 ones and composing a ten from 10 ones.

The power of the base-ten system is in repeated bundling by ten: 10 tens make a unit called a hundred. Repeating this process of creating new units by bundling in groups of ten creates units called *thousand*, *ten thousand*, *hundred thousand* .... In learning about decimals, children partition a one into 10 equal-sized smaller units, each of which is a tenth. Each base-ten unit can be understood in terms of any other base-ten unit. For example, one hundred can be viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths. Algorithms for operations in base ten draw on such relationships among the base-ten units.

Computations Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit.

Beginning in Kindergarten, the requisite abilities develop gradually over the grades. Experience with addition and subtraction within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2 standard.[2.OA.2] Computations within 20 that "cross 10," such as $9+8$ or $13-6$, are especially relevant to NBT because they afford the development of the Level 3 make-a-ten strategies for addition and subtraction described in the OA Progression. From the NBT perspective, make-a-ten strategies are (implicitly) the first instances of composing or decomposing a base-ten unit. Such strategies are a foundation for understanding in Grade 1 that addition may require composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve decomposing a ten.[2.NBT.7]

Strategies and algorithms The Standards distinguish strategies from algorithms.• For example, students use strategies for addition and subtraction in Grades K–3, but are expected to fluently add and subtract whole numbers using standard algorithms by the end of Grade 4. Use of the standard algorithms can be viewed as the culmination of a long progression of reasoning about quantities, the base-ten system, and the properties of operations.

This progression distinguishes between two types of computational strategies: special strategies and general methods. For example, a special strategy for computing $398+17$ is to decompose 17 as $2+15$, and evaluate $(398+2)+15$. Special strategies either cannot be extended to all numbers represented in the base-ten system or require considerable modification in order to do so. A more readily generalizable method of computing $398+17$ is to combine like base-ten units. General methods extend to all numbers represented in the base-ten system. A general method is not necessarily efficient. For example, counting on by ones is a general method that can be easily modified for use with finite decimals. General methods based on place value, however, are more efficient and can be viewed as closely connected with standard algorithms.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

2.OA.2 Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

• **Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.
Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Mathematical practices Both general methods and special strategies are opportunities to develop competencies relevant to the NBT standards. Use and discussion of both types of strategies offer opportunities for developing fluency with place value and properties of operations, and to use these in justifying the correctness of computations (MP.3). Special strategies may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, they offer opportunities to raise the topic of using appropriate tools strategically (MP.5). Standard algorithms can be viewed as expressions of regularity in repeated reasoning (MP.8) used in general methods based on place value.

Numerical expressions and recordings of computations, whether with strategies or standard algorithms, afford opportunities for students to contextualize, probing into the referents for the symbols involved (MP.2). Representations such as bundled objects or math drawings (e.g., drawings of hundreds, tens, and ones) and diagrams (e.g., simplified renderings of arrays or area models) afford the mathematical practice of explaining correspondences among different representations (MP.1). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten? This progression gives examples of representations that can be used to connect numerals with quantities and to connect numerical representations with combination, composition, and decomposition of base-ten units as students work towards computational fluency.

Kindergarten

In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1+9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place valueK.NBT.1 Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, . . . , nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section.

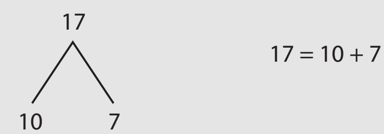
The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones ." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones).

By working with teen numbers in this way in Kindergarten, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

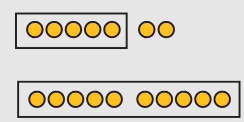
• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

Number-bond diagram and equation



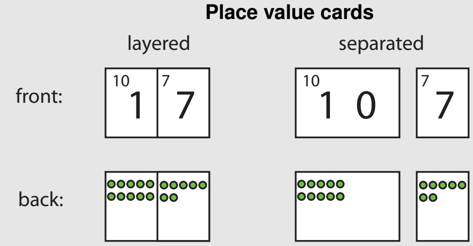
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames



Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

Place value cards



Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings.

Grade 1

In first grade, students learn to view ten ones as a unit called a ten. The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and they add and subtract using this understanding.

Extend the counting sequence and understand place value Through practice and structured learning time, students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers $10, \ldots, 90$, in written and in spoken form, as 1 ten, ..., 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones.

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

a 10 can be thought of as a bundle of ten ones—called a "ten."

b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols $>$, $=$, and $<$. Correctly placing the $<$ and $>$ symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.

1.NBT.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

Use place value understanding and properties of operations to add and subtract First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones. In particular, showing composition of a ten with objects or drawings

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

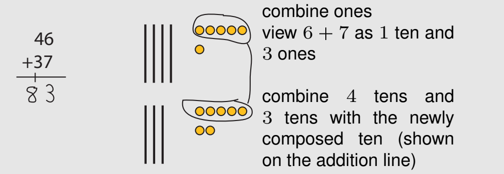
Adding tens and ones separately as illustrated in the margin is a general method that can extend to any sum of multi-digit numbers. Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings of 5-groups can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as $70-40$ can be viewed as 7 tens minus 4 tens and represented with concrete models such as objects bundled in tens or drawings. Children use the relationship between subtraction and addition when they view $80-70$ as an unknown addend addition problem, $70+\square=80$, and reason that 1 ten must be added to 70 to make 80, so $80-70=10$.

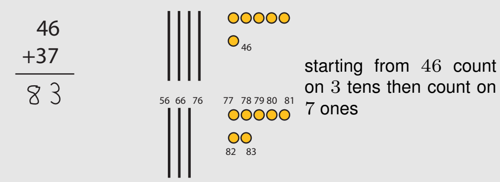
First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases.

General method: Adding tens and ones separately



This method is an application of the associative property.

Special strategy: Counting on by tens



This strategy requires counting on by tens from 46, beginning 56, 66, 76, then counting on by ones.

1.NBT.5 Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

1.NBT.6 Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

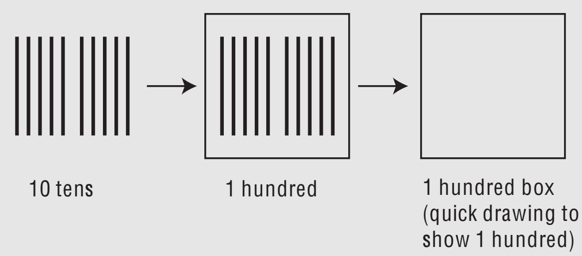
At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

2.NBT.1a Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a 100 can be thought of as a bundle of ten tens—called a "hundred."

Drawings to support seeing 10 tens as 1 hundred



Representations such as manipulative materials, math drawings and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3]

2.NBT.3 Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number reached after 500 counts of 1.

Students begin to work towards multiplication when they skip count by 5s, by 10s, and by 100s. This skip counting is not yet true multiplication because students don't keep track of the number of groups they have counted.[2.NBT.2]

2.NBT.2 Count within 1000; skip-count by 5s, 10s, and 100s.

Comparing magnitudes of two-digit numbers draws on the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers draws on the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4]

2.NBT.4 Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, =, and $<$ symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract Students become fluent in two-digit addition and subtraction.[2.NBT.5, 2.NBT.6] Representations such as manipulative materials and drawings may be used to support reasoning and explanations about addition and subtraction with three-digit numbers.[2.NBT.7] When students add ones to ones, tens to tens, and hundreds to hundreds they are implicitly using a general method based on place

2.NBT.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

2.NBT.6 Add up to four two-digit numbers using strategies based on place value and properties of operations.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

value and the associative and commutative properties of addition. Examples of how general methods can be represented in numerical work and composition and decomposition can be represented in math drawings are shown in the margin.

Drawings and diagrams can illustrate the reasoning repeated in general methods for computation that are based on place value. These provide an opportunity for students to observe this regularity and build toward understanding the standard addition and subtraction algorithms required in Grade 4 as expressions of repeated reasoning (MP.8).

At Grade 2, composing and decomposing involves an extra layer of complexity beyond that of Grade 1. This complexity manifests itself in two ways. First, students must understand that a hundred is a unit composed of 100 ones, but also that it is composed of 10 tens. Second, there is the possibility that both a ten and a hundred are composed or decomposed. For example, in computing $398 + 7$ a new ten and a new hundred are composed. In computing $302 - 184$, a ten and a hundred are decomposed.

Students may continue to develop and use special strategies for particular numerical cases or particular problem situations such as Unknown Addend. For example, instead of using a general method to add $398 + 7$, students could reason mentally by decomposing the 7 ones as $2 + 5$, adding 2 ones to 398 to make 400, then adding the remaining 5 ones to make 405. This method uses the associative property of addition and extends the make-a-ten strategy described in the OA Progression. Or students could reason that 398 is close to 400, so the sum is close to $400 + 7$, which is 407, but this must be 2 too much because 400 is 2 more than 398, so the actual sum is 2 less than 407, which is 405. Both of these strategies make use of place value understanding and are practical in limited cases.

Subtractions such as $302 - 184$ can be computed using a general method by decomposing a hundred into 10 tens, then decomposing one of those tens into 10 ones. Students could also view it as an unknown addend problem $184 + \square = 302$, thus drawing on the relationship between subtraction and addition. With this view, students can solve the problem by adding on to 184: first add 6 to make 190, then add 10 to make 200, next add 100 to make 300, and finally add 2 to make 302. They can then combine what they added on to find the answer to the subtraction problem: $6 + 10 + 100 + 2 = 118$. This strategy is especially useful in unknown addend situations. It can be carried out more easily in writing because one does not have to keep track of everything mentally. This is a Level 3 strategy, and is easier than the Level 3 strategy illustrated below that requires keeping track of how much of the second addend has been added on. (See the OA Progression for further discussion of levels.)

When computing sums of three-digit numbers, students might use strategies based on a flexible combination of Level 3 composition and decomposition and Level 2 counting-on strategies when finding the value of an expression such as $148 + 473$. For exam-

Addition: Recording combined hundreds, tens, and ones on separate lines

$$\begin{array}{r} 456 \\ +167 \\ \hline \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline 500 \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline 500 \\ 110 \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline 500 \\ 110 \\ 13 \\ \hline 623 \end{array}$$

Addition proceeds from left to right, but could also have gone from right to left. There are two advantages of working left to right: Many students prefer it because they read from left to right, and working first with the largest units yields a closer approximation earlier.

Addition: Recording newly composed units on the same line

$$\begin{array}{r} 456 \\ +167 \\ \hline \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline {}^{1}\ \\ 3 \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline {}^{1\,1}\ \ \\ 23 \end{array} \qquad \begin{array}{r} 456 \\ +167 \\ \hline {}^{1\,1}\ \ \\ 623 \end{array}$$

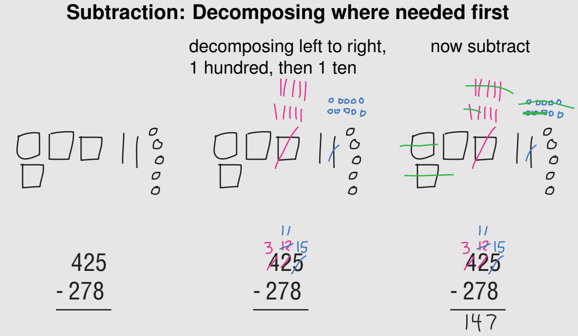
Add the ones, $6 + 7$, and record these 13 ones with 3 in the ones place and 1 on the line under the tens column.

Add the tens, $5 + 6 + 1$, and record these 12 tens with 2 in the tens place and 1 on the line under the hundreds column.

Add the hundreds, $4 + 1 + 1$ and record these 6 hundreds in the hundreds column.

Digits representing newly composed units are placed below the addends. This placement has several advantages. Each two-digit partial sum (e.g., "13") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 3) rather than "write the 3 and carry the 1" (write 3, then 1).

Subtraction: Decomposing where needed first



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

ple, they might say, "100 and 400 is 500. And 70 and 30 is another hundred, so 600. Then 8, 9, 10, 11 ... and the other 10 is 21. So, 621." Keeping track of what is being added is easier using a written form of such reasoning and makes it easier to discuss. There are two kinds of decompositions in this strategy. Both addends are decomposed into hundreds, tens, and ones, and the first addend is decomposed successively into the part already added and the part still to add.

Students should continue to develop proficiency with mental computation. They mentally add 10 or 100 to a given number between 100 and 900, and mentally subtract 10 or 100 from a given number between 100 and 900.[2.NBT.8]

[2.NBT.8] Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

Grade 3

At Grade 3, the major focus is multiplication,• so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

• See the progression on Operations and Algebraic Thinking.

Use place value understanding and properties of operations to perform multi-digit arithmetic Students continue adding and subtracting within 1000.[3.NBT.2] They achieve fluency with strategies and algorithms that are based on place value, properties of operations, and/or the relationship between addition and subtraction. Such fluency can serve as preparation for learning standard algorithms in Grade 4, if the computational methods used can be connected with those algorithms.

[3.NBT.2] Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students. Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

[3.NBT.1] Use place value understanding to round whole numbers to the nearest 10 or 100.

The special role of 10 in the base-ten system is important in understanding multiplication of one-digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication: $3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150$. It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non-zero digits, then shift the product one place to the left to make the result ten times as large.•

[3.NBT.3] Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

• **Grade 3 explanations for "15 tens is 150"**

- Skip-counting by 50. 5 tens is 50, 100, 150.
- Counting on by 5 tens. 5 tens is 50, 5 more tens is 100, 5 more tens is 150.
- Decomposing 15 tens. 15 tens is 10 tens and 5 tens. 10 tens is 100. 5 tens is 50. So 15 tens is 100 and 50, or 150.
- Decomposing 15.

$$\begin{aligned} 15 \times 10 &= (10 + 5) \times 10 \\ &= (10 \times 10) + (5 \times 10) \\ &= 100 + 50 \\ &= 150 \end{aligned}$$

All of these explanations are correct. However, skip-counting and counting on become more difficult to use accurately as numbers become larger, e.g., in computing 5×90 or explaining why 45 tens is 450, and needs modification for products such as 4×90. The first does not indicate any place value understanding.

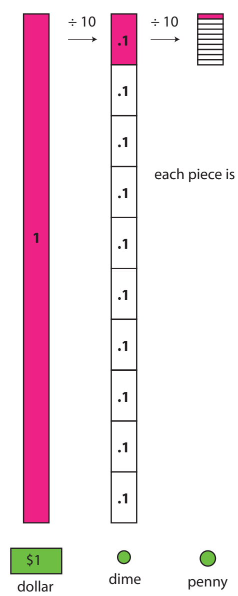
Grade 4

At Grade 4, students extend their work in the base-ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi-digit whole numbers In the base-ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between $1{,}000$ and $1{,}000{,}000$, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (thousand, million, billion, trillion, etc.). Thus, $457{,}000$ is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in previous grades apply to these numbers, because of the uniformity of the base-ten system.

Decimal notation and fractions Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base-ten system to non-whole numbers.

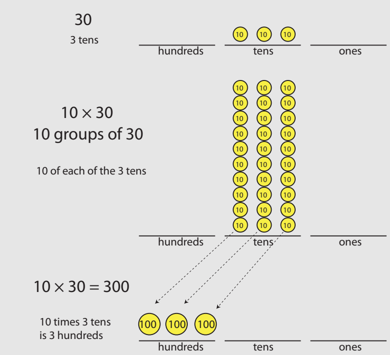


Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$.[4.NF.4b] As with whole-number expansions in the base-ten system, each unit in this decomposition is ten times the unit to its right. This can be connected with the use of base-ten notation to represent $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

4.NBT.1 Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 represented as 3 tens each taken 10 times

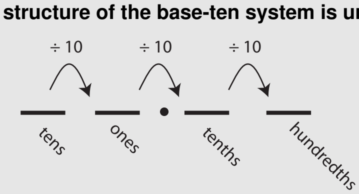


Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

4.NBT.2 Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using $>$, =, and $<$ symbols to record the results of comparisons.

4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[2]

The structure of the base-ten system is uniform



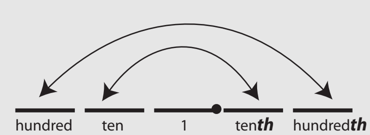
4.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, =, or $<$, and justify the conclusions, e.g., by using a visual model.

Symmetry with respect to the ones place



However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins $3.1415\ldots$.)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students become fluent with decimal equivalencies such as three tenths is thirty hundredths.

Use place value understanding and properties of operations to perform multi-digit arithmetic At Grade 4, students become fluent with the standard addition and subtraction algorithms.[4.NBT.4] As discussed at the beginning of this progression, these algorithms rely on adding or subtracting like base-ten units (ones with ones, tens with tens, hundreds with hundreds, and so on) and composing or decomposing base-ten units as needed (such as composing 10 ones to make 1 ten or decomposing 1 hundred to make 10 tens). In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable.

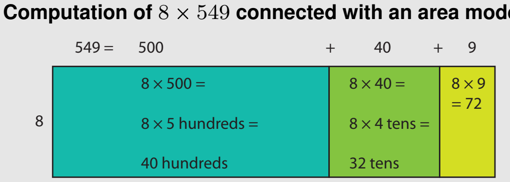
In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose. By reasoning repeatedly about the connection between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

Students can invent and use fast special strategies while also working towards understanding general methods and the standard algorithm.

[4.NBT.4] Fluently add and subtract multi-digit whole numbers using the standard algorithm.

[4.NBT.5] Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Computation of 8×549 connected with an area model



Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

This can also viewed as finding how many objects are in 8 groups of 549 objects, by finding the cardinalities of 8 groups of 500, 8 groups of 40, and 8 groups of 9, then adding them.

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into like base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

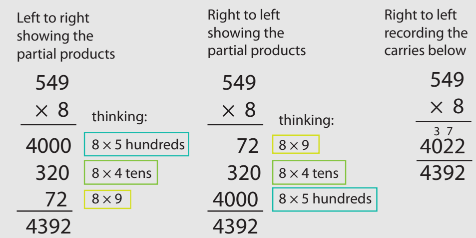
Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$\begin{aligned} 36 \times 94 &= (30+6) \times (90+4) \\ &= (30+6) \times 90 + (30+6) \times 4 \\ &= 30 \times 90 + 6 \times 90 + 30 \times 4 + 6 \times 4. \end{aligned}$$

General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

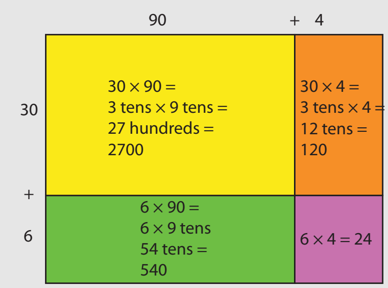
Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. As with multiplication, this relies on the distributive property. This can be viewed as finding the side length of a rectangle (the divisor is the length of the other side) or as allocating objects (the divisor is the number of groups). See the figures on the next page for examples.

Computation of 8×549: Ways to record general methods



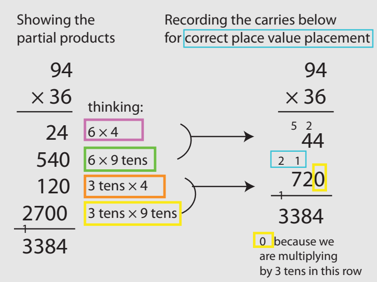
The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from $8 \times 9 = 72$ is written diagonally to the left of the 2 rather than above the 4 in 549.

Computation of 36×94 connected with an area model



The products of like base-ten units are shown as parts of a rectangular region.

Computation of 36×94: Ways to record general methods



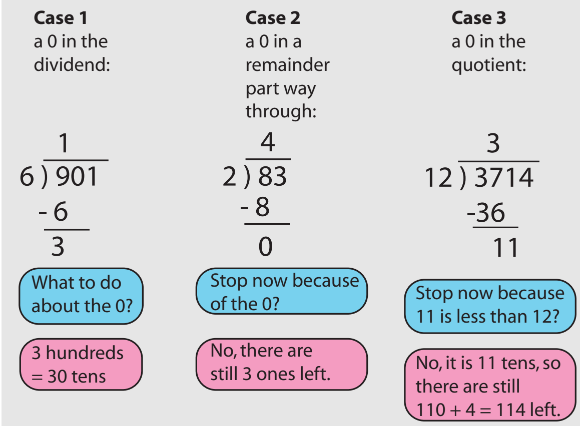
These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. The digits 2 and 1 are surrounded by a blue box. The 1 from $30 \times 4 = 120$ is placed correctly in the hundreds place and the digit 2 from $30 \times 90 = 2700$ is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 (surrounded by a yellow box) in the ones place of the second line of the method on the right is there because the whole line of digits is produced by multiplying by 30 (not 3).

[4.NBT.6] Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

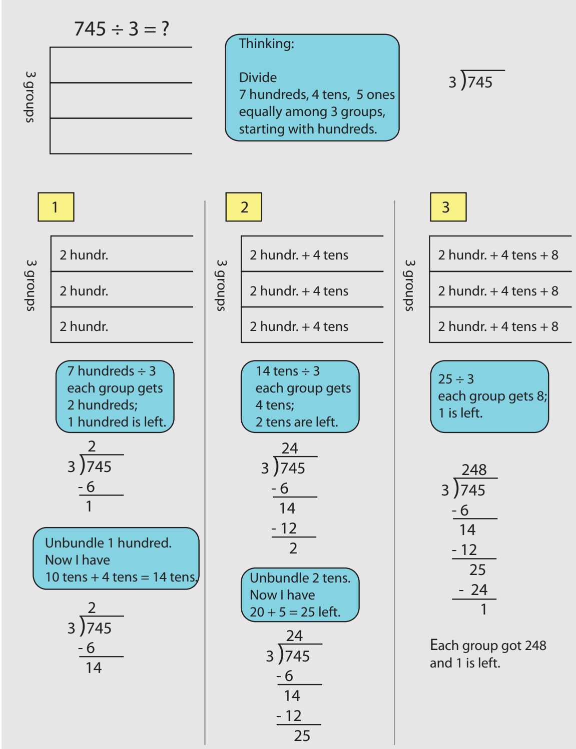
Multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$. In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation.

Cases involving 0 in division may require special attention.

Cases involving 0 in division

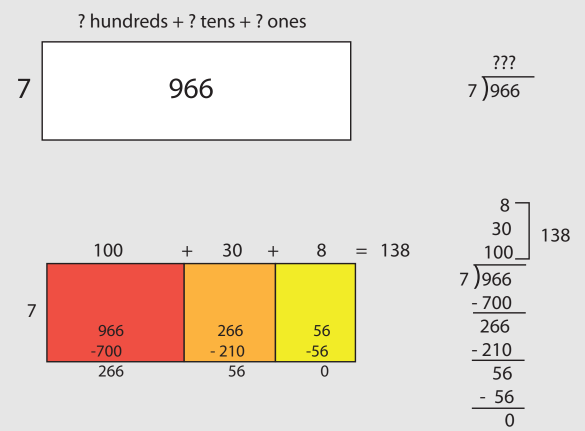


Division as finding group size



$745 \div 3$ can be viewed as allocating 745 objects bundled in 7 hundreds, 4 tens, and 3 ones equally among 3 groups. In Step 1, the 2 indicates that each group got 2 hundreds, the 6 is the number of hundreds allocated, and the 1 is the number of hundreds not allocated. After Step 1, the remaining hundred is decomposed as 10 tens and combined with the 4 tens (in 745) to make 14 tens.

Division as finding side length



$966 \div 7$ is viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The amount of hundreds is found, then tens, then ones. This yields a decomposition into three regions of dimensions 7 by 100, 7 by 30, and 7 by 8. It can be connected with the decomposition of 966 as $7 \times 100 + 7 \times 30 + 7 \times 8$. By the distributive property, this is $7 \times (100 + 30 + 8)$, so the unknown side length is 138. In the recording on the right, amounts of hundreds, tens, and ones are represented by numbers rather than by digits, e.g., 700 instead of 7.

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare, and how numbers round for decimals to thousandths.

New at Grade 5 is the use of whole number exponents to denote powers of 10.[5.NBT.2] Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right. So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole numbers and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value. Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

5.NBT.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Perform operations with multi-digit whole numbers and with decimals to hundredths At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

5.NBT.5 Fluently multiply multi-digit whole numbers using the standard algorithm.

Division strategies in Grade 5 involve breaking the dividend apart into like base-ten units and applying the distributive property to find the quotient place by place, starting from the highest place. (Division can also be viewed as finding an unknown factor: the dividend is the product, the divisor is the known factor, and the quotient is the unknown factor.) Students continue their fourth grade work on division, extending it to computation of whole number quotients with dividends of up to four digits and two-digit divisors. Estimation becomes relevant when extending to two-digit divisors. Even if students round appropriately, the resulting estimate may need to be adjusted.

Recording division after an underestimate

1655 ÷ 27		1	
		10	61
Rounding 27	(30)	50	
to 30 produces	27)	1655	
the underestimate		-1350	
50 at the first step		305	
but this method		-270	
allows the division		35	
process to be		-27	
continued		8	

Because of the uniformity of the structure of the base-ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.[5.NBT.7] Like base-ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. Although whole numbers are not usually written with a decimal point, but that a decimal point with 0s on its right can be inserted (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base-ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing new units below on the addition or subtraction line, and they can decompose units wherever needed before subtracting.

5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single-digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base-ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.[5.NF.3] When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.[5.NF.4] Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation-based method is not reliable in all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

5.NF.7b Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[3]

b Interpret division of a whole number by a unit fraction, and compute such quotients.

5.NF.5 Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Extending beyond Grade 5

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. Students use these ideas again when they calculate with polynomials in high school.

The distributive property and like units: Multiplication of whole numbers and polynomials

52×73	$(5x+2)(7x+3)$	
$= (5 \times 10 + 2)(7 \times 10 + 3)$	$= (5x+2)(7x+3)$	decomposing as like units (powers of 10 or powers of x)
$= 5 \times 10(7 \times 10 + 3) + 2 \times (7 \times 10 + 3)$	$= 5x(7x+3) + 2(7x+3)$	using the distributive property
$= 35 \times 10^2 + 15 \times 10 + 14 \times 10 + 2 \times 3$	$= 35x^2 + 15x + 14x + 2 \times 3$	using the distributive property again
$= 35 \times 10^2 + 29 \times 10 + 6$	$= 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)

3–5, Number and Operations – Fractions

Progressions for the Common Core State Standards in Mathematics (draft)*

12 August 2011

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[0]This document can be read with Preview on a Mac or with the latest version of Adobe Acrobat on a Mac or PC. It does not work with earlier versions of Acrobat.

3–5 Number and Operations—Fractions

Overview

Overview to be written.

Note. Changes such as including relevant equations or replacing with tape diagrams or fraction strips are planned for some diagrams. Some readers may find it helpful to create their own equations or representations.

Grade 3

The meaning of fractions In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.[2.G.3] In Grade 3 they start to develop the idea of a fraction more formally, building on the idea of partitioning a whole into equal parts. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision and measurement. In Grade 4, this is extended to include wholes that are collections of objects.

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

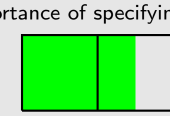
Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is the quantity you get by putting 3 of the $\frac{1}{4}$'s together.[3.NF.1] They read any fraction this way, and in particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is the quantity you get by combining 5 parts together when the whole is divided into 3 equal parts.

[3.NF.1] Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP6):

- Specifying the whole.
- Explaining what is meant by "equal parts."

The importance of specifying the whole



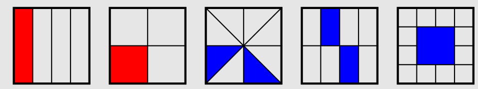
Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3).

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

Area representations of $\frac{1}{4}$



In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

The number line and number line diagrams On the number line, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler.

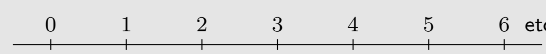
To construct a unit fraction on a number line diagram, e.g. $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They locate the number $\frac{1}{3}$ on the number line by marking off this length from 0, and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.[3.NF.2]

Students sometimes have difficulty perceiving the unit on a number line diagram. When locating a fraction on a number line diagram, they might use as the unit the entire portion of the number line that is shown on the diagram, for example indicating the number 3 when asked to show $\frac{3}{4}$ on a number line diagram marked from 0 to 4. Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, in the early stages of the NF Progression they use other representations such as area models, tape diagrams, and strips of paper. These, like number line diagrams, can be subdivided, representing an important aspect of fractions.

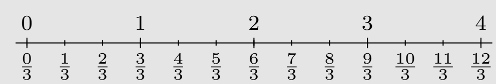
The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions, in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are

The number line



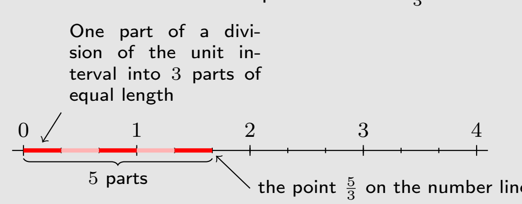
The number line marked off in thirds



[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

- a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
- b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Number line representation of $\frac{5}{3}$



therefore equal; that is, they are *equivalent fractions*. For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$ and also to $\frac{3}{6}$. Students can also use fraction strips to see fraction equivalence.[3.NF.3ab]

In particular, students in Grade 3 see whole numbers as fractions, recognizing, for example, that the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc. so that[3.NF.3c]

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots .$$

Of particular importance are the ways of writing 1 as a fraction:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots .$$

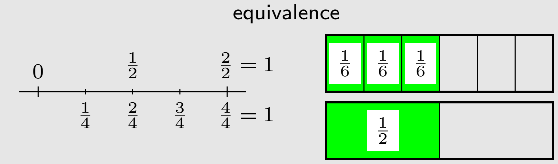
3.NF.3abc Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

Using the number line and fraction strips to see fraction equivalence



Comparing fractions Previously, in Grade 2, students compared lengths using a standard measurement unit.[2.MD.3] In Grade 3 they build on this idea to compare fractions with the same denominator. They see that for fractions that have the same denominator, the underlying unit fractions are the same size, so the fraction with the greater numerator is greater because it is made of more unit fractions. For example, segment from 0 to $\frac{3}{4}$ is shorter than the segment from 0 to $\frac{5}{4}$ because it measures 3 units of $\frac{1}{4}$ as opposed to 5 units of $\frac{1}{4}$. Therefore $\frac{3}{4} < \frac{5}{4}$.[3.NF.3d]

Students also see that for unit fractions, the one with the larger denominator is smaller, by reasoning, for example, that in order for more (identical) pieces to make the same whole, the pieces must be smaller. From this they reason that for fractions that have the same numerator, the fraction with the smaller denominator is greater. For example, $\frac{2}{5} > \frac{2}{7}$, because $\frac{1}{7} < \frac{1}{5}$, so 2 lengths of $\frac{1}{7}$ is less than 2 lengths of $\frac{1}{5}$.

As with equivalence of fractions, it is important in comparing fractions to make sure that each fraction refers to the same whole.

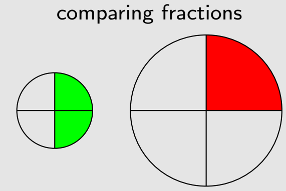
As students move towards thinking of fractions as points on the number line, they develop an understanding of order in terms of position. Given two fractions—thus two points on the number line—the one to the left is said to be smaller and the one to right is said to be larger. This understanding of order as position will become important in Grade 6 when students start working with negative numbers.

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

3.NF.3d Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

The importance of referring to the same whole when comparing fractions



A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

5

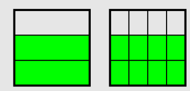
Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction. This property forms the basis for much of their other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

Equivalent fractions Students can use area models and number line diagrams to reason about equivalence.[4.NF.1] They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, n, corresponds physically to partitioning each unit fraction piece into n smaller equal pieces. The whole is then partitioned into n times as many pieces, and there are n times as many smaller unit fraction pieces as in the original fraction.

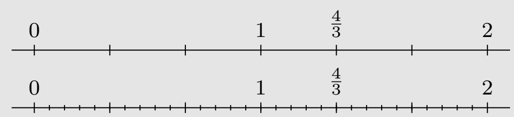
4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area model to show that $\frac{2}{3} = \frac{4\times 2}{4\times 3}$



The whole is the square, measured by area. On the left it is divided horizontally into 3 rectangles of equal area, and the shaded region is 2 of these and so represents $\frac{2}{3}$. On the right it is divided into 4×3 small rectangles of equal area, and the shaded area comprises 4×2 of these, and so it represents $\frac{4\times 2}{4\times 3}$.

Using the number line to show that $\frac{4}{3} = \frac{5\times 4}{5\times 3}$



$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5\times 3}$. Divide each of the intervals of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5\times 4}{5\times 3} = \frac{20}{15}$.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction as a point on the number line.

The fundamental property can be presented in terms of division, as in, e.g.

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}.$$

Because the equations $28 \div 4 = 7$ and $36 \div 4 = 9$ tell us that $28 = 4 \times 7$ and $36 = 4 \times 9$, this is the fundamental fact in disguise:

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

It is possible to over-emphasize the importance of simplifying fractions in this way. There is no mathematical reason why fractions must be written in simplified form, although it may be convenient to do so in some cases.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.[4.NF.2] For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they rewrite both fractions as

$$\frac{60}{96} \left(= \frac{12 \times 5}{12 \times 8}\right) \quad \text{and} \quad \frac{56}{96} \left(= \frac{7 \times 8}{12 \times 8}\right)$$

4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is therefore to the left of 1) but $\frac{13}{12}$ is greater than 1 (and is therefore to the right of 1).

Grade 5 students who have learned about fraction multiplication can see equivalence as "multiplying by 1":

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}$$

However, although a useful mnemonic device, this does not constitute a valid argument at this grade, since students have not yet learned fraction multiplication.

Adding and subtracting fractions The meaning of addition is the same for both fractions and whole numbers, even though algorithms for calculating their sums can be different. Just as the sum of 4 and 7 can be seen as the length of the segment obtained by joining together two segments of lengths 4 and 7, so the sum of $\frac{2}{3}$ and $\frac{8}{5}$ can be seen as the length of the segment obtained joining together two segments of length $\frac{2}{3}$ and $\frac{8}{5}$. It is not necessary to know how much $\frac{2}{3} + \frac{8}{5}$ is exactly in order to know what the sum means. This is analogous to understanding 51×78 as 51 groups of 78, without necessarily knowing its exact value.

This simple understanding of addition as putting together allows students to see in a new light the way fractions are built up from unit fractions. The same representation that students used in Grade 4 to see a fraction as a point on the number line now allows them to see a fraction as a sum of unit fractions: just as $5 = 1+1+1+1+1$, so

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

because $\frac{5}{3}$ is the total length of 5 copies of $\frac{1}{3}$.[4.NF.3]

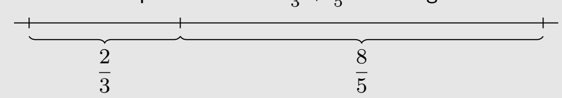
Armed with this insight, students decompose and compose fractions with the same denominator. They add fractions with the same denominator:[4.NF.3c]

$$\begin{aligned} \frac{7}{5} + \frac{4}{5} &= \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{7} + \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{4} \\ &= \frac{\overbrace{1+1+\cdots+1}^{7+4}}{5} \\ &= \frac{7+4}{5}. \end{aligned}$$

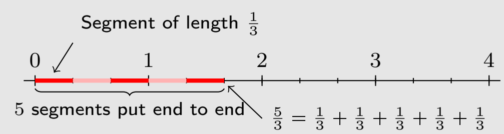
Using the understanding gained from work with whole numbers of the relationship between addition and subtraction, they also subtract fractions with the same denominator. For example, to subtract $\frac{5}{6}$ from $\frac{17}{6}$, they decompose

$$\frac{17}{6} = \frac{12}{6} + \frac{5}{6}, \quad \text{so} \quad \frac{17}{6} - \frac{5}{6} = \frac{17-5}{6} = \frac{12}{6} = 2.$$

Representation of $\frac{2}{3} + \frac{8}{5}$ as a length



Using the number line to see that $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$



[4.NF.3]Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators.• Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as a case of fraction addition.

Similarly, converting an improper fraction to a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.[4.NF.3b] Students can draw on their knowledge from Grade 3 of whole numbers as fractions. For example, knowing that $1 = \frac{3}{3}$, they see

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

Repeated reasoning with examples that gain in complexity leads to a general method involving the Grade 4 NBT skill of finding quotients and remainders.[4.NBT.6] For example,

$$\frac{47}{6} = \frac{(7\times 6)+5}{6} = \frac{7\times 6}{6} + \frac{5}{6} = 7 + \frac{5}{6} = 7\frac{5}{6}.$$

When solving word problems students learn to attend carefully to the underlying unit quantities. In order to formulate an equation of the form $A+B=C$ or $A-B=C$ for a word problem, the numbers A, B, and C must all refer to the same (or equivalent) wholes or unit amounts.[4.NF.3d] For example, students understand that the problem

> Bill had $\frac{2}{3}$ of a cup of juice. He drank $\frac{1}{2}$ of his juice. How much juice did Bill have left?

cannot be solved by subtracting $\frac{2}{3} - \frac{1}{2}$ because the $\frac{2}{3}$ refers to a cup of juice, but the $\frac{1}{2}$ refers to the amount of juice that Bill had, and not to a cup of juice. Similarly, in solving

> If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{3} + \frac{1}{4}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 3×7 can be represented as the number of objects in 3 groups of 7 objects, and write this as $7+7+7$. Grade 4 students apply this understanding to fractions, seeing

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \quad \text{as} \quad 5\times\frac{1}{3}.$$

• A mixed number is a whole number plus a fraction smaller than 1, written without the + sign, e.g. $5\frac{3}{4}$ means $5+\frac{3}{4}$ and $7\frac{1}{5}$ means $7+\frac{1}{5}$.

4.NF.3b Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

4.NF.3d Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

In general, they see a fraction as the numerator times the unit fraction with the same denominator,[4.NF.4a] e.g.,

$$\frac{7}{5} = 7 \times \frac{1}{5}, \qquad \frac{11}{3} = 11 \times \frac{1}{3}.$$

The same thinking, based on the analogy between fractions and whole numbers, allows students to give meaning to the product of a whole number and a fraction,[4.NF.4b] e.g., they see

$$3 \times \frac{2}{5} \quad \text{as} \quad \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}.$$

Students solve word problems involving multiplication of a fraction by a whole number.[4.NF.4c]

> If a bucket holds $2\frac{3}{4}$ gallons and 43 buckets of water fill a tank, how much does the tank hold?

The answer is $43 \times 2\frac{3}{4}$ gallons, which is

$$43 \times \left(2 + \frac{3}{4}\right) = 43 \times \frac{11}{4} = \frac{473}{4} = 118\frac{1}{4} \text{ gallons}$$

Decimals Fractions with denominator 10 and 100, called *decimal fractions*, arise naturally when student convert between dollars and cents, and have a more fundamental importance, developed in Grade 5, in the base 10 system. For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a concrete context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

Grade 3 students learn to add decimal fractions by converting them to fractions with the same denominator, in preparation for general fraction addition in Grade 5:[4.NF.5]

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Fractions with denominators equal to 10, 100, etc., such

$$\frac{27}{10}, \quad \frac{27}{100}, \quad \text{etc.}$$

can be written by using a *decimal point* as[4.NF.6]

$$2.7, \quad 0.27.$$

The number of digits to the right of the decimal point indicates the number of zeros in the denominator, so that $2.70 = \frac{270}{100}$ and

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

a Understand a fraction a/b as a multiple of $1/b$.

b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

[4.NF.5] Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[1]

[4.NF.6] Use decimal notation for fractions with denominators 10 or 100.

$2.7 = \frac{27}{10}$. Students use their ability to convert fractions to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

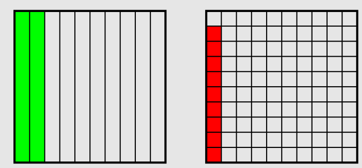
Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of them as 0.20 and 0.09 and see that $0.20 > 0.09$ because[4.NF.7]

$$\frac{20}{100} > \frac{9}{100}.$$

The argument using the meaning of a decimal as a fraction generalizes to work with decimals in Grade 5 that have more than two digits, whereas the argument using a visual fraction model, shown in the margin, does not. So it is useful for Grade 4 students to see such reasoning.

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$ using a visual fraction model



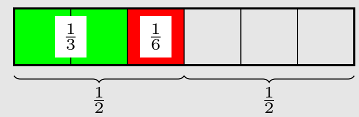
The shaded region on the left shows 0.2 of the unit square, since it is two parts when the square is divided int 10 parts of equal area. The shaded region on the right shows 0.09 of the unit square, since it is 9 parts when the unit is divided into 100 parts of equal area.

Grade 5

Adding and subtracting fractions In Grade 4, students calculate sums of fractions with different denominators where one denominator is a divisor of the other, so that only one fraction has to be changed. For example, they might have used a fraction strip to reason that

$$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2},$$

Using a fraction strip to show that $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$



and in working with decimals they added fractions with denominators 10 and 100, such as

$$\frac{2}{10}+\frac{7}{100}=\frac{20}{100}+\frac{7}{100}=\frac{27}{100}.$$

They understand the process as expressing both summands in terms of the same unit fraction so that they can be added. Grade 5 students extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.[5.NF.1] For example, in calculating $\frac{2}{3}+\frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is subdivided into fourths, and if each fourth in $\frac{5}{4}$ is subdivided into thirds, then each fraction will be a sum of unit fractions with denominator $3\times 4=4\times 3=12$:

$$\frac{2}{3}+\frac{5}{4}=\frac{2\times 4}{3\times 4}+\frac{5\times 3}{4\times 3}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}.$$

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

In general two fractions can be added by subdividing the unit fractions in one using the denominator of the other:

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}=\frac{ad+bc}{bd}.$$

It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.[5.NF.2] For example in the problem

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

> Ludmilla and Lazarus each have a lemon. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ a cup from hers and Lazarus squeezes $\frac{2}{5}$ of a cup from his. How much lemon juice to they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5}<\frac{1}{2}$. They calculate $\frac{1}{2}+\frac{2}{5}=\frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{2}{5}+\frac{2}{5}=\frac{3}{7}$ by noticing that $\frac{3}{7}<\frac{1}{2}$.

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, so that each person gets $50 \times \frac{1}{9} = \frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any quantity by a fraction.[5.NF.4a] Just as

$$\frac{1}{3} \times 5 \text{ is one part when } 5 \text{ is partitioned into } 3 \text{ parts,}$$

so

$$\frac{4}{3} \times 5 \text{ is } 4 \text{ parts when } 5 \text{ is partitioned into } 3 \text{ parts.}$$

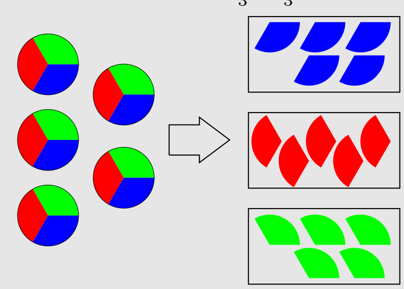
Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:
$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

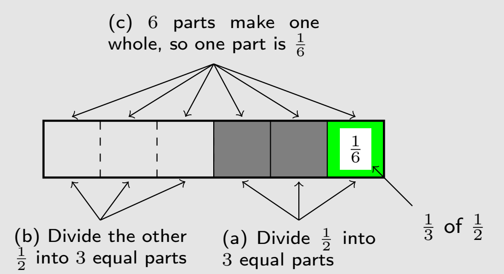


If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

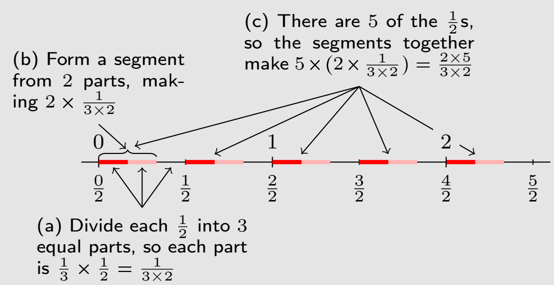
5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2\times5}{3\times2}$



For more complicated examples, an area model is useful, in which students work with a rectangle that has fractional side lengths, dividing it up into rectangles whose sides are the corresponding unit fractions.

Students also understand fraction multiplication by creating story contexts. For example, to explain

$$\frac{2}{3} \times 4 = \frac{8}{3},$$

they might say

> Ron and Hermione have 4 pounds of Bertie Bott's Every Flavour Beans. They decide to share them 3 ways, saving one share for Harry. How many pounds of beans do Ron and Hermione get?

Using the relationship between division and multiplication, students start working with simple fraction division problems. Having seen that division of a whole number by a whole number, e.g. $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that[5.NF.7a]

$$\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}.$$

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be 3×6 in 3, and so[5.NF.7b]

$$3 \div \frac{1}{6} = 3 \times 6 = 18.$$

Students use story problems to make sense of division problems:[5.NF.7c]

> How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?

Students attend carefully to the underlying unit quantities when solving problems. For example, if $\frac{1}{2}$ of a fund-raiser's funds were raised by the 6th grade, and if $\frac{1}{3}$ of the 6th grade's funds were raised by Ms. Wilkin's class, then $\frac{1}{3} \times \frac{1}{2}$ gives the fraction of the fund-raiser's funds that Ms. Wilkin's class raised, but it does not tell us how much money Ms. Wilkin's class raised.[5.NF.6]

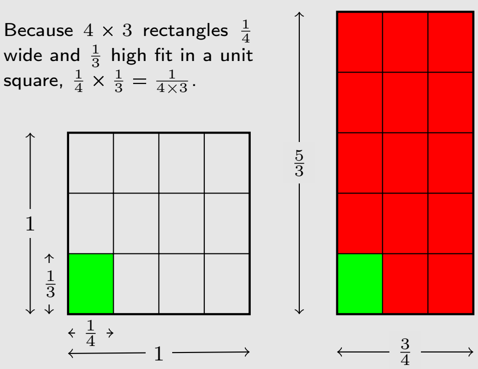
Multiplication as scaling In preparation for Grade 6 work in ratios and proportional reasoning, students learn to see products such as 5×3 or $\frac{1}{2} \times 3$ as expressions that can be interpreted in terms of a quantity, 3, and a scaling factor, 5 or $\frac{1}{2}$. Thus, in addition to knowing that $5 \times 3 = 15$, they can also say that 5×3 is 5 times as big as 3, without evaluating the product. Likewise, they see $\frac{1}{2} \times 3$ as half the size of 3.[5.NF.5a]

5.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3\times5}{4\times3}$

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high fit in a unit square, $\frac{1}{4} \times \frac{1}{3} = \frac{1}{4\times3}$.



The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4\times3}$, so has area $\frac{3\times5}{4\times3}$.

5.NF.7abc Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[2]

a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.5a Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

The understanding of multiplication as scaling is an important opportunity for students to reasonin abstractly (MP2). Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a recipe is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when the budget of a large state university is multiplied by $\frac{1}{2}$, for example.[5.NF.5b]

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 6.

5.NF.5b Interpret multiplication as scaling (resizing), by:

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

K–6, Geometry

Progressions for the Common Core State Standards in Mathematics (draft)

23 June 2012

K–6, Geometry

Overview

Like core knowledge of number, core geometrical knowledge appears to be a universal capability of the human mind. Geometric and spatial thinking are important in and of themselves, because they connect mathematics with the physical world, and play an important role in modeling phenomena whose origins are not necessarily physical, for example, as networks or graphs. They are also important because they support the development of number and arithmetic concepts and skills. Thus, geometry is essential for all grade levels for many reasons: its mathematical content, its roles in physical sciences, engineering, and many other subjects, and its strong aesthetic connections.

This progression discusses the most important goals for elementary geometry according to three categories.

- Geometric shapes, their components (e.g., sides, angles, faces), their properties, and their categorization based on those properties.

- Composing and decomposing geometric shapes.

- Spatial relations and spatial structuring.

Geometric shapes, components, and properties. Students develop through a series of levels of geometric and spatial thinking. As with all of the domains discussed in the Progressions, this development depends on instructional experiences. Initially, students cannot reliably distinguish between examples and nonexamples of categories of shapes, such as triangles, rectangles, and squares.• With experience, they progress to the next level of thinking, recognizing shapes in ways that are visual or syncretic (a fusion of differing systems). At this level, students can recognize shapes as wholes, but cannot form mathematically-constrained mental images of them. A given figure is a rectangle, for example, because "it looks like a door." They do not explicitly think about the components or about the defining attributes, or properties, of shapes. Students then move to a descriptive level in which they can think about the components of shapes, such as triangles having three sides. For example, kindergartners can decide whether all of the sides of a shape

• In formal mathematics, a geometric shape is a boundary of a region, e.g., "circle" is the boundary of a disk. This distinction is not expected in elementary school.

are straight and they can count the sides. They also can discuss if the shape is closed• and thus convince themselves that a three-sided shape is a triangle even if it is "very skinny" (e.g., an isosceles triangle with large obtuse angle).

At the analytic level, students recognize and characterize shapes by their *properties*.[1] For instance, a student might think of a square as a figure that has four equal sides and four right angles. Different components of shapes are the focus at different grades, for instance, second graders measure lengths and fourth graders measure angles (see the Geometric Measurement Progression). Students find that some combinations of properties signal certain classes of figures and some do not; thus the seeds of geometric implication are planted. However, only at the next level, abstraction, do students see relationships between classes of figures (e.g., understand that a square is a rectangle because it has all the properties of rectangles).• Competence at this level affords the learning of higher-level geometry, including deductive arguments and proof.

Thus, learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years.

Composing and decomposing. As with their learning of shapes, components, and properties, students follow a progression to learn about the composition and decomposition of shapes. Initially, they lack competence in composing geometric shapes. With experience, they gain abilities to combine shapes into pictures—first, through trial and error, then gradually using attributes. Finally, they are able to synthesize combinations of shapes into new shapes.•

Students compose new shapes by putting two or more shapes together and discuss the shapes involved as the parts and the totals. They decompose shapes in two ways. They take away a part by covering the total with a part (for example, covering the "top" of a triangle with a smaller triangle to make a trapezoid). And they take shapes apart by building a copy beside the original shape to see what shapes that shape can be decomposed into (initially, they may need to make the decomposition on top of the total shape). With

[1] In this progression, the term "property" is reserved for those attributes that indicate a relationship between components of shapes. Thus, "having parallel sides" or "having all sides of equal lengths" are properties. "Attributes" and "features" are used interchangeably to indicate any characteristic of a shape, including properties, and other defining characteristics (e.g., straight sides) and nondefining characteristics (e.g., "right-side up").

• A shape with straight sides is closed if exactly two sides meet at every vertex, every side meets exactly two other sides, and no sides cross each other.

Levels of geometric thinking

Visual/syncretic. Students recognize shapes, e.g., a rectangle "looks like a door."

Descriptive. Students perceive properties of shapes, e.g., a rectangle has four sides, all its sides are straight, opposite sides have equal length.

Analytic. Students characterize shapes by their properties, e.g., a rectangle has opposite sides of equal length and four right angles.

Abstract. Students understand that a rectangle is a parallelogram because it has all the properties of parallelograms.

• Note that in the U.S., that the term "trapezoid" may have two different meanings. In their study *The Classification of Quadrilaterals* (Information Age Publishing, 2008), Usiskin et al. call these the exclusive and inclusive definitions:

T(E): a trapezoid is a quadrilateral with exactly one pair of parallel sides

T(I): a trapezoid is a quadrilateral with at least one pair of parallel sides.

These different meanings result in different classifications at the analytic level. According to T(E), a parallelogram is not a trapezoid; according to T(I), a parallelogram is a trapezoid.

Both definitions are legitimate. However, Usiskin et al. conclude, "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-bound geometry books, to favor the inclusive definition."

• **A note about research** The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis. Additionally, there is suggestive evidence that this type of composition corresponds with, and may support, children's ability to compose and decompose numbers.

experience, students are able to use a composed shape as a new unit in making other shapes. Grade 1 students make and use such a unit of units (for example, making a square or a rectangle from two identical right triangles, then making pictures or patterns with such squares or rectangles). Grade 2 students make and use three levels of units (making an isosceles triangle from two 1″ by 2″ right triangles, then making a rhombus from two of such isosceles triangles, and then using such a rhombus with other shapes to make a picture or a pattern). Grade 2 students also compose with two such units of units (for example, making adjacent strips from a shorter parallelograms made from a 1″ by 2″ rectangle and two right triangles and a longer parallelogram made from a 1″ by 3″ parallelogram and the same two right triangles). Grade 1 students also rearrange a composite shape to make a related shape, for example, they change a 1″ by 2″ rectangle made from two right triangles into an isosceles triangle by flipping one right triangle. They explore such rearrangements of the two right triangles more systematically by matching the short right angle side (a tall isosceles triangle and a parallelogram with a "little slant"), then the long right angle sides (a short isosceles triangle and a parallelogram with a "long slant"). Grade 2 students rearrange more complex shapes, for example, changing a parallelogram made from a rectangle and two right triangles into a trapezoid by flipping one of the right triangles to make a longer and a shorter parallel side.

Composing and decomposing requires and thus builds experience with properties such as having equal lengths or equal angles.

Spatial structuring and spatial relations. Early composition and decomposition of shape is a foundation for spatial structuring, an important case of geometric composition and decomposition. Students need to conceptually structure an array to understand two-dimensional objects and sets of such objects in two-dimensional space as truly two-dimensional. Such spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because it takes previously abstracted items as content and integrates them to form new structures. For two-dimensional arrays, students must see a composite of squares (iterated units) and as a composite of rows or columns (units of units). Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane. Spatial relations such as above/below and right/left are understood within such spatial structures. These understandings begin informally, later becoming more formal.

The ability to structure a two-dimensional rectangular region into rows and columns of squares requires extended experiences with shapes derived from squares (e.g., squares, rectangles, and right triangles) and with arrays of contiguous squares that form patterns.

Development of this ability benefits from experience with compositions, decompositions, and iterations of the two, but it requires extensive experience with arrays.

Students make pictures from shapes whose sides or points touch, and they fill in outline puzzles. These gradually become more elaborate, and students build mental visualizations that enable them to move from trial and error rotating of a shape to planning the orientation and moving the shape as it moves toward the target location. Rows and columns are important units of units within square arrays for the initial study of area, and squares of 1 by 1, 1 by 10, and 10 by 10 are the units, units of units, and units of units of units used in area models of two-digit multiplication in Grade 4. Layers of three-dimensional shapes are central for studying volume in Grade 5. Each layer of a right rectangular prism can also be structured in rows and columns, such layers can also be viewed as units of units of units. That is, as 1000 is a unit (one thousand) of units (one hundred) of units (tens) of units (singletons), a right rectangular prism can be considered a unit (solid, or three-dimensional array) of units (layers) of units (rows) of units (unit cubes).

Summary. The Standards for Kindergarten, Grade 1, and Grade 2 focus on three major aspects of geometry. Students build understandings of shapes and their properties, becoming able to do and discuss increasingly elaborate compositions, decompositions, and iterations of the two, as well as spatial structures and relations. In Grade 2, students begin the formal study of measure, learning to use units of length and use and understand rulers. Measurement of angles and parallelism are a focus in Grades 3, 4, and 5. At Grade 3, students begin to consider relationships of shape categories, considering two levels of subcategories (e.g., rectangles are parallelograms and squares are rectangles). They complete this categorization in Grade 5 with all necessary levels of categories and with the understanding that any property of a category also applies to all shapes in any of its subcategories. They understand that some categories overlap (e.g., not all parallelograms are rectangles) and some are disjoint (e.g., no square is a triangle), and they connect these with their understanding of categories and subcategories. Spatial structuring for two- and three-dimensional regions is used to understand what it means to measure area and volume of the simplest shapes in those dimensions: rectangles at Grade 3 and right rectangular prisms at Grade 5 (see the Geometric Measurement Progression).

K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/“corners”) and other attributes (e.g., having sides of equal length).

Kindergarden

Understanding and describing shapes and space is one of the two critical areas of Kindergarten mathematics. Students develop geometric concepts and spatial reasoning from experience with two perspectives on space: the shapes of objects and the relative positions of objects.

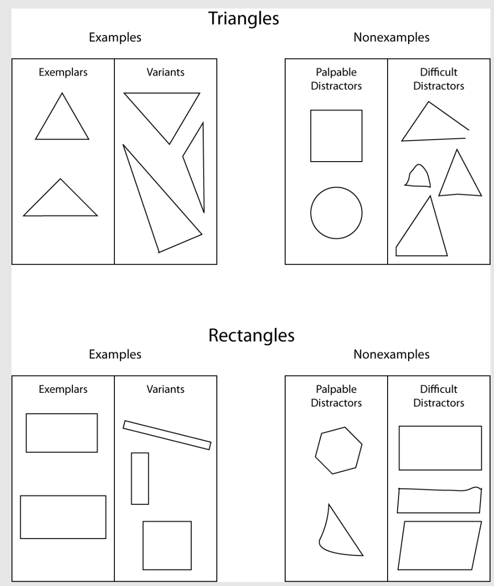
In the domain of shape, students learn to match two-dimensional shapes even when the shapes have different orientations.[K.G.4] They learn to name shapes such as circles, triangles, and squares, whose names occur in everyday language, and distinguish them from nonexamples of these categories, often based initially on visual prototypes. For example, they can distinguish the most typical examples of triangles from the obvious nonexamples.

From experiences with varied examples of these shapes (e.g., the variants shown in the margin), students extend their initial intuitions to increasingly comprehensive and accurate intuitive concept images of each shape category.• These richer concept images support students' ability to perceive a variety of shapes in their environments and describe these shapes in their own words.[MP7] This includes recognizing and informally naming three-dimensional shapes, e.g., "balls," "boxes," "cans." Such learning might also occur in the context of solving problems that arise in construction of block buildings and in drawing pictures, simple maps, and so forth.

Students then refine their informal language by learning mathematical concepts and vocabulary so as to increasingly describe their physical world from geometric perspectives, e.g., shape, orientation, spatial relations (MP4). They increase their knowledge of a variety of shapes, including circles, triangles, squares, rectangles, and special cases of other shapes such as regular hexagons, and trapezoids with unequal bases and non-parallel sides of equal length.[K.G.1] • They learn to sort shapes according to these categories.[MP7] The need to explain their decisions about shape names or classifications prompts students to attend to and describe certain features of the shapes.[K.G.4] That is, concept images and names they have learned for the shapes are the raw material from which they can abstract common features.[MP2] This also supports their learning to represent shapes informally with drawings and by building them from components (e.g., manipulatives such as sticks).[K.G.5] With repeated experiences such as these, students become more precise (MP6). They begin to attend to attributes, such as being a triangle, square, or rectangle, and being *closed* figures with *straight* sides. Similarly, they attend to the lengths of sides and, in simple situations, the size of angles when comparing shapes.

Students also begin to name and describe three-dimensional shapes with mathematical vocabulary, such as "sphere," "cube," "cylinder," and "cone."[K.G.1] They identify faces of three-dimensional shapes as two-dimensional geometric figures[K.G.4] and explicitly identify shapes as two-dimensional ("flat" or lying in a plane) or three-dimensional

[K.G.4] Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).



Exemplars are the typical visual prototypes of the shape category.

Variants are other examples of the shape category.

Palpable distractors are nonexamples with little or no overall resemblance to the exemplars.

Difficult distractors are visually similar to examples but lack at least one defining attribute.

• Tall and Vinner describe *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (See "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics, 12*, pp. 151–169.) This term was formulated by Shlomo Vinner in 1980.

[MP7] Mathematically proficient students look closely to discern a pattern or structure.

[K.G.1] Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

• If the exclusive definition of trapezoid is used (see p. 3), such trapezoids would be called isosceles trapezoids.

[MP7] Young students, for example, . . . may sort a collection of shapes according to how many sides the shapes have.

[MP2] Mathematically proficient students have the ability to abstract a given situation.

[K.G.5] Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

("solid").K.G.3

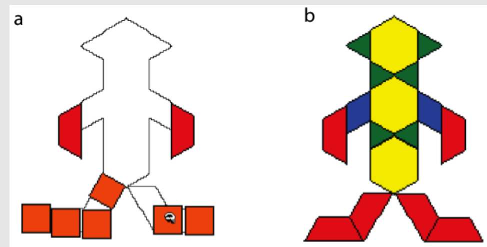
A second important area for kindergartners is the composition of geometric figures. Students not only build shapes from components, but also compose shapes to build pictures and designs. Initially lacking competence in composing geometric shapes, they gain abilities to combine shapes–first by trial and error and gradually by considering components–into pictures. At first, side length is the only component considered. Later experience brings an intuitive appreciation of angle size.

Students combine two-dimensional shapes and solve problems such as deciding which piece will fit into a space in a puzzle, intuitively using geometric motions (slides, flips, and turns, the informal names for translations, reflections, and rotations, respectively). They can construct their own outline puzzles and exchange them, solving each other's.

Finally, in the domain of spatial reasoning, students discuss not only shape and orientation, but also the relative positions of objects, using terms such as "above," "below," "next to," "behind," "in front of," and "beside."K.G.1 They use these spatial reasoning competencies, along with their growing knowledge of three-dimensional shapes and their ability to compose them, to model objects in their environment, e.g., building a simple representation of the classroom using unit blocks and/or other solids (MP4).

K.G.3 Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Combining shapes to build pictures



Students first use trial and error (part a) and gradually consider components (part b).

Grade 1

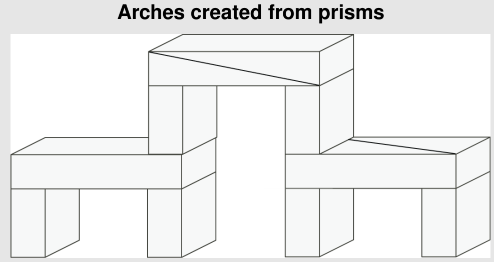
In Grade 1, students reason about shapes. They describe and classify shapes, including drawings, manipulatives, and physical-world objects, in terms of their geometric *attributes*. That is, based on early work recognizing, naming, sorting, and building shapes from components, they describe in their own words why a shape belongs to a given category, such as squares, triangles, circles, rectangles, rhombuses, (regular) hexagons, and trapezoids (with bases of different lengths and nonparallel sides of the same length). In doing so, they differentiate between geometrically defining attributes (e.g., "hexagons have six straight sides") and nondefining attributes (e.g., color, overall size, or orientation).[1.G.1] For example, they might say of this shape, "This has to go with the squares, because all four sides are the same, and these are square corners. It doesn't matter which way it's turned" (MP3, MP7). They explain why the variants shown earlier (p. 6) are members of familiar shape categories and why the difficult distractors are not, and they draw examples and nonexamples of the shape categories. Students learn to sort shapes accurately and exhaustively based on these attributes, describing the similarities and differences of these familiar shapes and shape categories (MP7, MP8).

[1.G.1] Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) ; build and draw shapes to possess defining attributes.

From the early beginnings of informally matching shapes and solving simple shape puzzles, students learn to intentionally compose and decompose plane and solid figures (e.g., putting two congruent isosceles triangles together with the explicit purpose of making a rhombus),[1.G.2] building understanding of part-whole relationships as well as the properties of the original and composite shapes. In this way, they learn to perceive a combination of shapes as a single new shape (e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles). Thus, they develop competencies that include solving shape puzzles and constructing designs with shapes, creating and maintaining a shape as a unit, and combining shapes to create composite shapes that are conceptualized as independent entities (MP2). They then learn to substitute one composite shape for another congruent composite composed of different parts.

[1.G.2] Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[2]

Arches created from prisms



Right rectangular prisms are composed with prisms with right triangle bases. Note that the dimensions of the triangular prism on the top arch differ from the dimensions of that on the right.

Students build these competencies, often more slowly, in the domain of three-dimensional shapes. For example, students may intentionally combine two right triangular prisms to create a right rectangular prism, and recognize that each triangular prism is half of the rectangular prism.[1.G.3] They also show recognition of the composite shape of "arch." (Note that the process of combining shapes to create a composite shape is much like combining 10 ones to make 1 ten.) Even simple compositions, such as building a floor or wall of rectangular prisms, build a foundation for later mathematics.

[1.G.3] Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

As students combine shapes, they continue to develop their sophistication in describing geometric attributes and properties and determining how shapes are alike and different, building foundations for measurement and initial understandings of properties such as congruence and symmetry. Students can learn to use their intuitive understandings of measurement, congruence, and symmetry to guide their work on tasks such as solving puzzles and making simple origami constructions by folding paper to make a given two- or three-dimensional shape (MP1).•

• For example, students might fold a square of paper once to make a triangle or nonsquare rectangle. For examples of other simple two- and three-dimensional origami constructions, see http://www.origami-instructions.com/simple-origami.html.

Grade 2

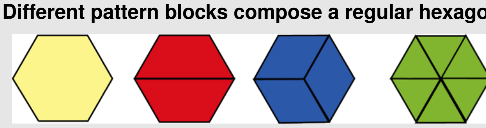
Students learn to name and describe the defining attributes of categories of two-dimensional shapes, including circles, triangles, squares, rectangles, rhombuses, trapezoids, and the general category of quadrilateral. They describe pentagons, hexagons, septagons, octagons, and other polygons by the number of sides, for example, describing a septagon as either a "seven-gon" or simply "seven-sided shape" (MP2).[2.G.1] Because they have developed both verbal descriptions of these categories and their defining attributes and a rich store of associated mental images, they are able to draw shapes with specified attributes, such as a shape with five sides or a shape with six angles.[2.G.1] They can represent these shapes' attributes accurately (within the constraints of fine motor skills). They use length to identify the properties of shapes (e.g., a specific figure is a rhombus because all four of its sides have equal length). They recognize right angles, and can explain the distinction between a rectangle and a parallelogram without right angles and with sides of different lengths (sometimes called a "rhomboid").

[2.G.1] Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[3] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

[2.G.1] Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[4] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

Students learn to combine their composition and decomposition competencies to build and operate on composite units (units of units), intentionally substituting arrangements or composites of smaller shapes or substituting several larger shapes for many smaller shapes, using geometric knowledge and spatial reasoning to develop foundations for area, fraction, and proportion. For example, they build the same shape from different parts, e.g., making with pattern blocks, a regular hexagon from two trapezoids, three rhombuses, or six equilateral triangles. They recognize that the hexagonal faces of these constructions have equal area, that each trapezoid has half of that area, and each rhombus has a third of that area.[2.G.3]

Different pattern blocks compose a regular hexagon

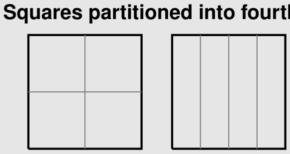


[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

This example emphasizes the fraction concepts that are developed; students can build and recognize more difficult composite shapes and solve puzzles with numerous pieces. For example, a tangram is a special set of 7 shapes which compose an isosceles right triangle. The tangram pieces can be used to make many different configurations and tangram puzzles are often posed by showing pictures of these configurations as silhouettes or outlines. These pictures often are made more difficult by orienting the shapes so that the sides of right angles are not parallel to the edges of the page on which they are displayed. Such pictures often do not show a grid that shows the composing shapes and are generally not preceded by analysis of the composing shapes.

Students also explore decompositions of shapes into regions that are congruent or have equal area.[2.G.3] For example, two squares can be partitioned into fourths in different ways. Any of these fourths represents an equal share of the shape (e.g., "the same amount of cake") even though they have different shapes.

Squares partitioned into fourths



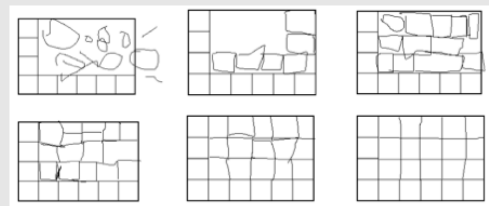
These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Another type of composition and decomposition is essential to students' mathematical development—*spatial structuring*. Students need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP7).

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Levels of thinking in spatial structuring

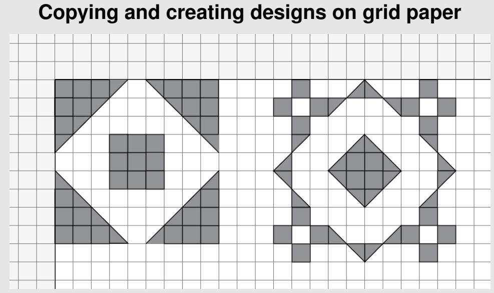


Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because previously abstracted items (e.g., squares, including composites made up of squares) are used as the content of new mental structures. Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. At first, students might tile a rectangle with identical squares or draw such arrays and then count the number of squares one-by-one. In the lowest levels of the progression, they may even lose count of or double-count some squares. As the mental structuring process helps them organize their counting, they become more systematic, using the array structure to guide the quantification. Eventually, they begin to use repeated addition of the number in each row or each column. Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane.

Foundational activities, such as forming arrays by tiling a rectangle with identical squares (as in building a floor or wall from blocks) should have developed students' basic spatial structuring competencies before second grade—if not, teachers should ensure that their students learn these skills. Spatial structuring can be further developed with several activities with grids. Games such as "battleship" can be useful in this regard.

Copying and creating designs on grid paper



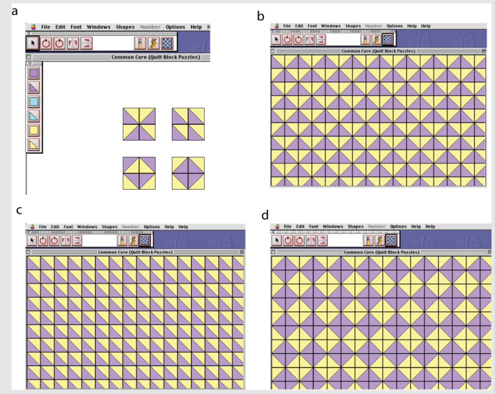
Students can copy designs such as these, using only squares (all of the same size) and isosceles right triangles (half of the square) as manipulatives, creating their copies on paper with grid squares of the same size as the manipulative square.

Another useful type of instructional activity is copying and creating designs on grids. Students can copy designs drawn on grid paper by placing manipulative squares and right triangles onto other copies of the grid. They can also create their own designs, draw their creations on grid paper, and exchange them, copying each others' designs.

Another, more complex, activity designing tessellations by iterating a "core square." Students design a unit composed of smaller units: a core square composed of a 2 by 2 array of squares filled with square or right triangular regions. They then create the tessellation ("quilt") by iterating that core in the plane. This builds spatial structuring because students are iterating "units of units" and reflecting on the resulting structures. Computer software can aid in this iteration.

These various types of composition and decomposition experiences simultaneously develop students' visualization skills, including recognizing, applying, and anticipating (MP1) the effects of applying rigid motions (slides, flips, and turns) to two-dimensional shapes.

"Core squares" iterated to make a tessellation



In the software environment illustrated above (Pattern Blocks and Mini-Quilts software), students need to be explicitly aware of the transformations they are using in order to use slide, flip, and turn tools. At any time, they can tessellate any one of the core squares using the "quilt" tool indicated by the rightmost icon. Part a shows four different core squares. The upper left core square produces the tessellation in part b. Parts c and d are produced, respectively, by the upper right and lower right core squares. Interesting discussions result when the class asks which designs are mathematically different (e.g., should a rotation or flip of the core be counted as "different"s?).

Grade 3

Students analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3).[3.G.1] They explicitly relate and combine these classifications. Because they have built a firm foundation of several shape categories, these categories can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories. A description of these categories of quadrilaterals is illustrated in the margin. The Standards do not require that such representations be constructed by Grade 3 students, but they should be able to draw examples of quadrilaterals that are not in the subcategories.

Similarly, students learn to draw shapes with prespecified attributes, without making a priori assumptions regarding their classification.[MP1] For example, they could solve the problem of making a shape with two long sides of the same length and two short sides of the same length that is not a rectangle.

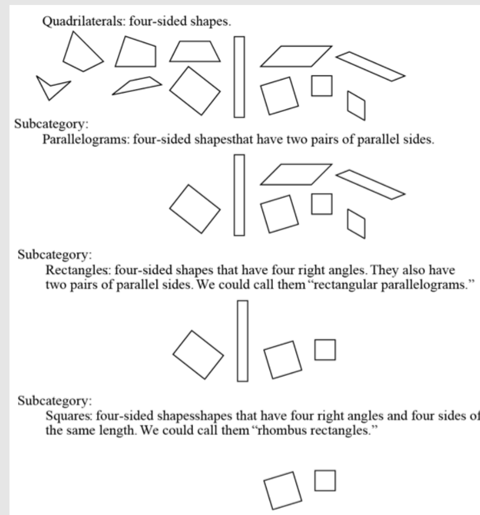
Students investigate, describe, and reason about decomposing and composing polygons to make other polygons. Problems such as finding all the possible different compositions of a set of shapes involve geometric problem solving and notions of congruence and symmetry (MP7). They also involve the practices of making and testing conjectures (MP1), and convincing others that conjectures are correct (or not) (MP3). Such problems can be posed even for sets of simple shapes such as tetrominoes, four squares arranged to form a shape so that every square shares at least one side and sides coincide or share only a vertex.

More advanced paper-folding (origami) tasks afford the same mathematical practices of seeing and using structure, conjecturing, and justifying conjectures. Paper folding can also illustrate many geometric concepts. For example, folding a piece of paper creates a line segment. Folding a square of paper twice, horizontal edge to horizontal edge, then vertical edge to vertical edge, creates a right angle, which can be unfolded to show four right angles. Students can be challenged to find ways to fold paper into rectangles or squares and to explain why the shapes belong in those categories.

Students also develop more competence in the composition and decomposition of rectangular regions, that is, spatially structuring rectangular arrays. They learn to partition a rectangle into identical squares[3.G.2] by anticipating the final structure and thus forming the array by drawing rows and columns (see the bottom right example on p. 11; some students may still need work building or drawing squares inside the rectangle first). They count by the number of columns or rows, or use multiplication to determine the number of

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

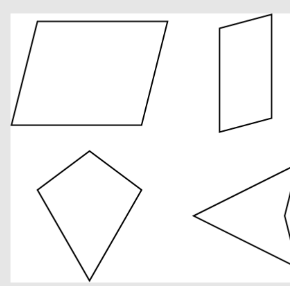
Quadrilaterals and some special kinds of quadrilaterals



The representations above might be used by teachers in class. Note that the left-most four shapes in the first section at the top left have four sides but do not have properties that would place them in any of the other categories shown (parallelograms, rectangles, squares).

[MP1] Students . . . analyze givens, constraints, relationships, and goals.

Quadrilaterals that are not rectangles



These quadrilaterals have two pairs of sides of the same length but are not rectangles. A kite is on lower left and a deltoid is at lower right.

[3.G.2] Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

squares in the array. They also learn to rotate these arrays physically and mentally to view them as composed of smaller arrays, allowing illustrations of properties of multiplication (e.g., the commutative property and the distributive property).

Grade 4

Students describe, analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3), including explicit use of angle sizes[4.G.1] and the related geometric properties of perpendicularity and parallelism.[4.G.2] They can identify these properties in two-dimensional figures. They can use side length to classify triangles as equilateral, equiangular, isosceles, or scalene; and can use angle size to classify them as acute, right, or obtuse. They then learn to cross-classify, for example, naming a shape as a right isosceles triangle. Thus, students develop explicit awareness of and vocabulary for many concepts they have been developing, including points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Such mathematical terms are useful in communicating geometric ideas, but more important is that constructing examples of these concepts, such as drawing angles and triangles that are acute, obtuse, and right,[4.G.1] help students form richer concept images connected to verbal definitions. That is, students have more complete and accurate mental images and associated vocabulary for geometric ideas (e.g., they understand that angles can be larger than 90° and their concept images for angles include many images of such obtuse angles). Similarly, students see points and lines as abstract objects: Lines are infinite in extent and points have location but no dimension. Grids are made of points and lines and do not end at the edge of the paper.

Students also learn to apply these concepts in varied contexts (MP4). For example, they learn to represent angles that occur in various contexts as two rays, explicitly including the reference line, e.g., a horizontal or vertical line when considering slope or a "line of sight" in turn contexts. They understand the size of the angle as a rotation of a ray on the reference line to a line depicting slope or as the "line of sight" in computer environments. Students might solve problems of drawing shapes with turtle geometry.• Analyzing the shapes in order to construct them (MP1) requires students to explicitly formulate their ideas about the shapes (MP4, MP6). For instance, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such experiences help students connect what are often initially isolated ideas about the concept of angle.

Students might explore line segments, lengths, perpendicularity, and parallelism on different types of grids, such as rectangular and

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

• The computer programming language Logo has a pointer, often a icon of a turtle, that draws representations of points, line segments, and shapes, with commands such as "forward 100" and "right 120."

triangular (isometric) grids (MP1, MP2).[4.G.2, 4.G.3] Can you find a non-rectangular parallelogram on a rectangular grid? Can you find a rectangle on a triangular grid? Given a segment on a rectangular grid that is not parallel to a grid line, draw a parallel segment of the same length with a given endpoint. Given a half of a figure and a line of symmetry, can you accurately draw the other half to create a symmetric figure?

Students also learn to reason about these concepts. For example, in "guess my rule" activities, they may be shown two sets of shapes and asked where a new shape belongs (MP1, MP2).[4.G.2]

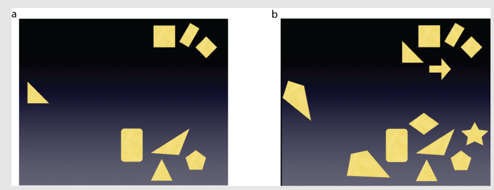
In an interdisciplinary lesson (that includes science and engineering ideas as well as items from mathematics), students might encounter another property that all triangles have: rigidity. If four fingers (both thumbs and index fingers) form a shape (keeping the fingers all straight), the shape of that quadrilateral can be easily changed by changing the angles. However, using three fingers (e.g., a thumb on one hand and the index and third finger of the other hand), students can see that the shape is fixed by the side lengths. Triangle rigidity explains why this shape is found so frequently in bridge, high-wire towers, amusement park rides, and other constructions where stability is sought.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

Guess My Rule



Students can be shown the two groups of shapes in part a and asked "Where does the shape on the left belong?" They might surmise that it belongs with the other triangles at the bottom. When the teacher moves it to the top, students must search for a different rule that fits all the cases.

Later (part b), students may induce the rule: "Shapes with at least one right angle are at the top." Students with rich visual images of right angles and good visualization skills would conclude that the shape at the left (even though it looks vaguely like another one already at the bottom) has one right angle, thus belongs at the top.

Grade 5

By the end of Grade 5, competencies in shape composition and decomposition, and especially the special case of spatial structuring of rectangular arrays (recall p. 11), should be highly developed (MP7). Students need to develop these competencies because they form a foundation for understanding multiplication, area, volume, and the coordinate plane. To solve area problems, for example, the ability to decompose and compose shapes plays multiple roles. First, students understand that the area of a shape (in square units) is the number of unit squares it takes to cover the shape without gaps or overlaps. They also use decomposition in other ways. For example, to calculate the area of an "L-shaped" region, students might decompose the region into rectangular regions, then decompose each region into an array of unit squares, spatially structuring each array into rows or columns. Students extend their spatial structuring in two ways. They learn to spatially structure in three dimensions; for example, they can decompose a right rectangular prism built from cubes into layers, seeing each layer as an array of cubes. They use this understanding to find the volumes of right rectangular prisms with edges whose lengths are whole numbers as the number of unit cubes that pack the prisms (see the Geometric Measurement Progression). Second, students extend their knowledge of the coordinate plane, understanding the continuous nature of two-dimensional space and the role of fractions in specifying locations in that space.

Thus, spatial structuring underlies coordinates for the plane as well, and students learn both to apply it and to distinguish the objects that are structured. For example, they learn to interpret the components of a rectangular grid structure as line segments or lines (rather than regions) and understand the precision of location that these lines require, rather than treating them as fuzzy boundaries or indicators of intervals. Students learn to reconstruct the levels of counting and quantification that they had already constructed in the domain of discrete objects to the coordination of (at first) two continuous linear measures. That is, they learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions.[5.G.1]

Although students can often "locate a point," these understandings are beyond simple skills. For example, initially, students often fail to distinguish between two different ways of viewing the point $(2, 3)$, say, as instructions: "right 2, up 3"; and as the point defined by being a distance 2 from the y-axis and a distance 3 from the x-axis. In these two descriptions the 2 is first associated with the x-axis, then with the y-axis.

They connect ordered pairs of (whole number) coordinates to points on the grid, so that these coordinate pairs constitute numerical objects and ultimately can be operated upon as single mathematical entities. Students solve mathematical and real-world problems using coordinates. For example, they plan to draw a symmetric fig-

[5.G.1] Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).

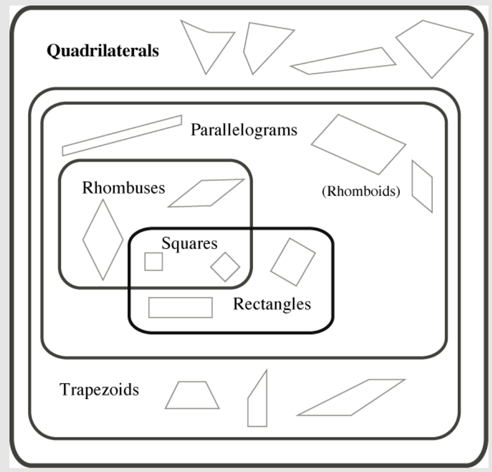
ure using computer software in which students' input coordinates that are then connected by line segments.[5.G.2]

Students learn to analyze and relate categories of two-dimensional and three-dimensional shapes explicitly based on their properties.[5.G.4] Based on analysis of properties, they classify two-dimensional figures in hierarchies. For example, they conclude that all rectangles are parallelograms, because they are all quadrilaterals with two pairs of opposite, parallel, equal-length sides (MP3). In this way, they relate certain categories of shapes as subclasses of other categories.[5.G.3] This leads to understanding propagation of properties; for example, students understand that squares possess all properties of rhombuses and of rectangles. Therefore, if they then show that rhombuses' diagonals are perpendicular bisectors of one another, they infer that squares' diagonals are perpendicular bisectors of one another as well.

[5.G.2] Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

[5.G.4] Classify two-dimensional figures in a hierarchy based on properties.

Venn diagram showing classification of quadrilaterals



Note that rhomboids are parallelograms that are not rhombuses or rectangles. This example uses the inclusive definition of trapezoid (see p. [pageref "T(E)"]).

[5.G.3] Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

Grade 6

Problems involving areas and volumes extend previous work and provide a context for developing and using equations.[6.G.1, 6.G.2] Students' competencies in shape composition and decomposition, especially with spatial structuring of rectangular arrays (recall p. 11), should be highly developed. These competencies form a foundation for understanding multiplication, formulas for area and volume, and the coordinate plane.[6.NS.6, 6.NS.8]

Using the shape composition and decomposition skills acquired in earlier grades, students learn to develop area formulas for parallelograms, then triangles. They learn how to address three different cases for triangles: a height that is a side of a right angle, a height that "lies over the base" and a height that is outside the triangle.[MP.1]

Through such activity, students learn that that any side of a triangle can be considered as a base and the choice of base determines the height (thus, the base is not necessarily horizontal and the height is not always in the interior of the triangle). The ability to view a triangle as part of a parallelogram composed of two copies of that triangle and the understanding that area is additive (see the Geometric Measurement Progression) provides a justification (MP3) for halving the product of the base times the height, helping students guard against the common error of forgetting to take half.

Also building on their knowledge of composition and decomposition, students decompose rectilinear polygons into rectangles, and decompose special quadrilaterals and other polygons into triangles and other shapes, using such decompositions to determine their areas, and justifying and finding relationships among the formulas for the areas of different polygons.

Building on the knowledge of volume (see the Geometric Measurement Progression) and spatial structuring abilities developed in earlier grades, students learn to find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism.[6.G.2 MP.1 MP.4]

Students also analyze and compose and decompose polyhedral solids. They describe the shapes of the faces, as well as the number of faces, edges, and vertices. They make and use drawings of solid shapes and learn that solid shapes have an outer surface as well as an interior. They develop visualization skills connected to their mathematical concepts as they recognize the existence of, and visualize, components of three-dimensional shapes that are not visible from a given viewpoint (MP1). They measure the attributes of these shapes, allowing them to apply area formulas to solve surface area problems (MP7). They solve problems that require them to distinguish between units used to measure volume and units used to measure area (or length). They learn to plan the construction of

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

MP.1 Students . . . try special cases and simpler forms of the original problem in order to gain insight into its solution.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

MP.1 explain correspondences

MP.4 write an equation to describe a situation.

complex three-dimensional compositions through the creation of corresponding two-dimensional nets (e.g., through a process of digital fabrication and/or graph paper).[6.G.4] For example, they may design a living quarters (e.g., a space station) consistent with given specifications for surface area and volume (MP2, MP7). In this and many other contexts, students learn to apply these strategies and formulas for areas and volumes to the solution of real-world and mathematical problems.[6.G.1, 6.G.2] These problems include those in which areas or volumes are to be found from lengths or lengths are to be found from volumes or areas and lengths.

Students extend their understanding of properties of two-dimensional shapes to use of coordinate systems.[6.G.3] For example, they may specify coordinates for a polygon with specific properties, justifying the attribution of those properties through reference to relationships among the coordinates (e.g., justifying that a shape is a parallelogram by computing the lengths of its pairs of horizontal and vertical sides).

As a precursor for learning to describe cross-sections of three-dimensional figures,[7.G.3] students use drawings and physical models to learn to identify parallel lines in three-dimensional shapes, as well as lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.

[6.G.4] Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

[6.G.1] Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[6.G.2] Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

[6.G.3] Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

[7.G.3] Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Where the Geometry Progression is Heading

Composition and decomposition of shapes is used throughout geometry from Grade 6 to high school and beyond. Compositions and decompositions of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems that involve complex shapes. Decompositions are often indicated in geometric diagrams by an auxiliary line, and using the strategy of drawing an auxiliary line to solve a problem are part of looking for and making use of structure (MP7). Recognizing the significance of an existing line in a figure is also part of looking for and making use of structure. This may involve identifying the length of an associated line segment, which in turn may rely on students' abilities to identify relationships of line segments and angles in the figure. These abilities become more sophisticated as students gain more experience in geometry. In Grade 7, this experience includes making scale drawings of geometric figures and solving problems involving angle measure, surface area, and volume (which builds on understandings described in the Geometric Measurement Progression as well as the ability to compose and decompose figures).

K–5, Geometric Measurement

Progressions for the Common Core State Standards in Mathematics (draft)

23 June 2012

K–5, Geometric Measurement

1

Overview

Geometric measurement connects the two most critical domains of early mathematics, geometry and number, with each providing conceptual support to the other. Measurement is central to mathematics, to other areas of mathematics (e.g., laying a sensory and conceptual foundation for arithmetic with fractions), to other subject matter domains, especially science, and to activities in everyday life. For these reasons, measurement is a core component of the mathematics curriculum.

Measurement is the process of assigning a number to a magnitude of some attribute shared by some class of objects, such as length, relative to a unit. Length is a *continuous* attribute—a length can always be subdivided in smaller lengths. In contrast, we can count 4 apples exactly—cardinality is a discrete attribute. We can add the 4 apples to 5 other apples and know that the result is exactly 9 apples. However, the *weight* of those apples is a continuous attribute, and scientific measurement with tools gives only an approximate measurement—to the nearest pound (or, better, kilogram) or the nearest 1/100$^{\text{th}}$ of a pound, but always with some error.•

• The Standards do not differentiate between weight and mass. Technically, mass is the amount of matter in an object. Weight is the force exerted on the body by gravity. On the earth's surface, the distinction is not important (on the moon, an object would have the same mass, would weight less due to the lower gravity).

Before learning to measure attributes, children need to recognize them, distinguishing them from other attributes. That is, the attribute to be measured has to "stand out" for the student and be discriminated from the undifferentiated sense of amount that young children often have, labeling greater lengths, areas, volumes, and so forth, as "big" or "bigger."

Students then can become increasingly competent at *direct comparison*—comparing the amount of an attribute in two objects without measurement. For example, two students may stand back to back to directly compare their heights. In many circumstances, such direct comparison is impossible or unwieldy. Sometimes, a third object can be used as an intermediary, allowing *indirect comparison*. For example, if we know that Aleisha is taller than Barbara and that

[1] This progression concerns Measurement and Data standards related to geometric measurement. The remaining Measurement and Data standards are discussed in the K–3 Categorical Data and Grades 2–5 Measurement Data Progressions.

Barbara is taller than Callie, then we know (due to the transitivity of "taller than") that Aleisha is taller than Callie, even if Aleisha and Callie never stand back to back.•

• "Transitivity" abbreviates the Transitivity Principle for Indirect Measurement stated in the Standards as:

> If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

The purpose of measurement is to allow indirect comparisons of objects' amount of an attribute using numbers. An attribute of an object is measured (i.e., assigned a number) by comparing it to an amount of that attribute held by another object. One measures length with length, mass with mass, torque with torque, and so on. In geometric measurement, a unit is chosen and the object is subdivided or partitioned by copies of that unit and, to the necessary degree of precision, units subordinate to the chosen unit, to determine the number of units and subordinate units in the partition.

Personal benchmarks, such as "tall as a doorway" build students' intuitions for amounts of a quantity and help them use measurements to solve practical problems. A combination of internalized units and measurement processes allows students to develop increasing accurate estimation competencies.

Both in measurement and in estimation, the concept of *unit* is crucial. The concept of basic (as opposed to subordinate) unit just discussed is one aspect of this concept. The basic unit can be informal (e.g., about a car length) or standard (e.g., a meter). The distinction and relationship between the notion of discrete "1" (e.g., one apple) and the continuous "1" (e.g., one inch) is important mathematically and is important in understanding number line diagrams (e.g., see Grade 2) and fractions (e.g., see Grade 3). However, there are also superordinate units or "units of units." A simple example is a kilometer consisting of 1,000 meters. Of course, this parallels the number concepts students must learn, as understanding that tens and hundreds are, respectively, "units of units" and "units of units of units" (i.e., students should learn that 100 can be simultaneously considered as 1 hundred, 10 tens, and 100 ones).

Students' understanding of an attribute that is measured with derived units is dependent upon their understanding that attribute as entailing other attributes simultaneously. For example,

- Area as entailing two lengths, simultaneously;
- Volume as entailing area and length (and thereby three lengths), simultaneously.

Scientists measure many types of attributes, from hardness of minerals to speed. This progression emphasizes the geometric attributes of length, area, and volume. Nongeometric attributes such as weight, mass, capacity, time, and color, are often taught effectively in science and social studies curricula and thus are not extensively discussed here. Attributes derived from two different attributes, such as speed (derived from distance and time), are discussed in the high school Number and Quantity Progression and in the 6–7 Ratio and Proportion Progression.

Length is a characteristic of an object found by quantifying how far it is between the endpoints of the object. "Distance" is often used similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects, choosing a unit of measure and *subdividing* (mentally and physically) the object by that unit, placing that unit end to end (*iterating*) alongside the object. The length of the object is the number of units required to iterate from one end of the object to the other, without gaps or overlaps.

Length is a core concept for several reasons. It is the basic geometric measurement. It is also involved in area and volume measurement, especially once formulas are used. Length and unit iteration are critical in understanding and using the number line in Grade 3 and beyond (see the Number and Operations—Fractions Progression). Length is also one of the most prevalent metaphors for quantity and number, e.g., as the master metaphor for magnitude (e.g., vectors, see the Number and Quantity Progression). Thus, length plays a special role in this progression.

Area is an amount of two-dimensional surface that is contained within a plane figure. Area measurement assumes that congruent figures enclose equal areas, and that area is *additive*, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas. Area is measured by tiling a region with a two-dimensional unit (such as a square) and parts of the unit, without gaps or overlaps. Understanding how to spatially structure a two-dimensional region is an important aspect of the progression in learning about area.

Volume is an amount of three-dimensional space that is contained within a three-dimensional shape. Volume measurement assumes that congruent shapes enclose equal volumes, and that volume is *additive*, i.e., the volume of the union of two regions that overlap only at their boundaries is the sum of their volumes. Volume is measured by packing (or tiling, or tessellating) a region with a three-dimensional unit (such as a cube) and parts of the unit, without gaps or overlaps. Volume not only introduces a third dimension and thus an even more challenging spatial structuring, but also complexity in the nature of the materials measured. That is, solid units might be "packed," such as cubes in a three-dimensional array or cubic meters of coal, whereas liquids "fill" three-dimensional regions, taking the shape of a container, and are often measured in units such as liters or quarts.

A final, distinct, geometric attribute is *angle measure*. The size of an angle is the amount of rotation between the two rays that form the angle, sometimes called the sides of the angles.

Finally, although the attributes that we measure differ as just described, it is important to note: *central characteristics of measurement are the same for all of these attributes.* As one more testament to these similarities, consider the following side-by-side comparison of the Standards for measurement of area in Grade 3 and the measurement of volume in Grade 5.

5

Grade 3	Grade 5

Understand concepts of area and relate area to multiplication and to addition.

3.MD.5. Recognize area as an attribute of plane figures and understand concepts of area measurement.

a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7. Relate area to the operations of multiplication and addition.

a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Understand concepts of volume and relate volume to multiplication and to addition.

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5.MD.5 Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Kindergarten

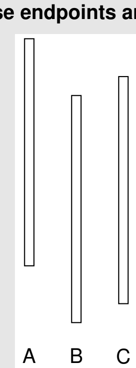
Describe and compare measurable attributes Students often initially hold undifferentiated views of measurable attributes, saying that one object is "bigger" than another whether it is longer, or greater in area, or greater in volume, and so forth. For example, two students might both claim their block building is "the biggest." Conversations about how they are comparing—one building may be taller (greater in length) and another may have a larger base (greater in area)—help students learn to discriminate and name these measureable attributes. As they discuss these situations and compare objects using different attributes, they learn to distinguish, label, and describe several measureable attributes of a single object.[K.MD.1] Thus, teachers listen for and extend conversations about things that are "big," or "small," as well as "long," "tall," or "high," and name, discuss, and demonstrate with gestures the attribute being discussed (length as extension in one dimension is most common, but area, volume, or even weight in others).

K.MD.1 Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

Length Of course, such conversations often occur in comparison situations ("He has more than me!"). Kindergartners easily directly compare lengths in simple situations, such as comparing people's heights, because standing next to each other automatically aligns one endpoint.[K.MD.2] However, in other situations they may initially compare only one endpoint of objects to say which is longer. Discussing such situations (e.g., when a child claims that he is "tallest" because he is standing on a chair) can help students resolve and coordinate perceptual and conceptual information when it conflicts. Teachers can reinforce these understandings, for example, by holding two pencils in their hand showing only one end of each, with the longer pencil protruding less. After asking if they can tell which pencil is longer, they reveal the pencils and discuss whether children were "fooled." The necessity of aligning endpoints can be explicitly addressed and then re-introduced in the many situations throughout the day that call for such comparisons. Students can also make such comparisons by moving shapes together to see which has a longer side.

K.MD.2 Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference.

Sticks whose endpoints are not aligned



When shown this figure and asked which is "the longest stick," students may point to A because it "sticks out the farthest." Similarly, they may recognize a 12-inch vertical line as "tall" and a 12-inch horizontal line as "long" but not recognize that the two are the same length.

Even when students seem to understand length in such activities, they may not conserve length. That is, they may believe that if one of two sticks of equal lengths is vertical, it is then longer than the other, horizontal, stick. Or, they may believe that a string, when bent or curved, is now shorter (due to its endpoints being closer to each other). Both informal and structured experiences, including demonstrations and discussions, can clarify how length is maintained, or conserved, in such situations. For example, teachers and students might rotate shapes to see its sides in different orientations. As with number, learning and using language such as "It looks longer, but it really isn't longer" is helpful.

Students who have these competencies can engage in experiences that lay the groundwork for later learning. Many can begin to learn to compare the lengths of two objects using a third object, order lengths, and connect number to length. For example, informal experiences such as making a road "10 blocks long" help students build a foundation for measuring length in the elementary grades. See the Grade 1 section on length for information about these important developments.

Area and volume Although area and volume experiences are not instructional foci for Kindergarten, they are attended to, at least to distinguish these attributes from length, as previously described. Further, certain common activities can help build students' experiential foundations for measurement in later grades. Understanding area requires understanding this attribute as the amount of two-dimensional space that is contained within a boundary. Kindergartners might informally notice and compare areas associated with everyday activities, such as laying two pieces of paper on top of each other to find out which will allow a "bigger drawing." Spatial structuring activities described in the Geometry Progression, in which designs are made with squares covering rectilinear shapes also help to create a foundation for understanding area.

Similarly, kindergartners might compare the capacities of containers informally by pouring (water, sand, etc.) from one to the other. They can try to find out which holds the most, recording that, for example, the container labeled "J" holds more than the container labeled "D" because when J was poured into D it overflowed. Finally, in play, kindergartners might make buildings that have layers of rectangular arrays. Teachers aware of the connections of such activities to later mathematics can support students' growth in multiple domains (e.g., development of self-regulation, social-emotional, spatial, and mathematics competencies) simultaneously, with each domain supporting the other.

Grade 1

Length comparisons First graders should continue to use direct comparison—carefully, considering all endpoints—when that is appropriate. In situations where direct comparison is not possible or convenient, they should be able to use indirect comparison and explanations that draw on transitivity (MP3). Once they can compare lengths of objects by direct comparison, they could compare several items to a single item, such as finding all the objects in the classroom the same length as (or longer than, or shorter than) their forearm.[1.MD.1] Ideas of transitivity can then be discussed as they use a string to represent their forear's length. As another example, students can figure out that one path from the teachers' desk to the door is longer than another because the first path is longer than a length of string laid along the path, but the other path is shorter than that string. Transitivity can then be explicitly discussed: If A is longer than B and B is longer than C, then A must be longer than C as well.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Seriation Another important set of skills and understandings is ordering a set of objects by length.[1.MD.1] Such sequencing requires multiple comparisons. Initially, students find it difficult to seriate a large set of objects (e.g., more than 6 objects) that differ only slightly in length. They tend to order groups of two or three objects, but they cannot correctly combine these groups while putting the objects in order. Completing this task efficiently requires a systematic strategy, such as moving each new object "down the line" to see where it fits. Students need to understand that each object in a seriation is larger than those that come before it, and shorter than those that come after. Again, reasoning that draws on transitivity is relevant.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Such seriation and other processes associated with the measurement and data standards are important in themselves, but also play a fundamental role in students' development. The general reasoning processes of seriation, conservation (of length and number), and classification (which lies at the heart of the standards discussed in the K–3 Categorical Data Progression) predict success in early childhood as well as later schooling.

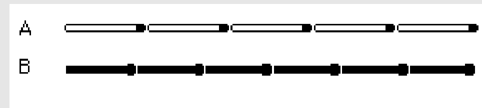
Measure lengths indirectly and by iterating length units Directly comparing objects, indirectly comparing objects, and ordering objects by length are important practically and mathematically, but they are not length measurement, which involves assigning a number to a length. Students learn to lay physical units such as centimeter or inch manipulatives end-to-end and count them to measure a length.[1.MD.2] Such a procedure may seem to adults to be straightforward, however, students may initially iterate a unit leaving gaps between subsequent units or overlapping adjacent units. For such students, measuring may be an activity of placing units along a

1.MD.2 Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

path in some manner, rather than the activity of covering a region or length with no gaps.

Also, students, especially if they lack explicit experience with continuous attributes, may make their initial measurement judgments based upon experiences counting discrete objects. For example, researchers showed children two rows of matches. The matches in each row were of different lengths, but there was a different number of matches in each so that the rows were the same length. Although, from the adult perspective, the lengths of the rows were the same, many children argued that the row with 6 matches was longer because it had more matches. They counted units (matches), assigning a number to a *discrete* attribute (cardinality). In measuring *continuous* attributes, the sizes of the units (white and dark matches) must be considered. First grade students can learn that objects used as basic units of measurement (e.g., "match-length") must be the same size.

Matches of different lengths



Row A is 5 matches long–when the unit of measurement is white matches. Row B is 6 matches long–when the unit of measurement is dark matches. From Inhelder, Sinclair, and Bovet, 1974, *Learning and the Development of Cognition*, *Harvard University Press.*

As with transitive reasoning tasks, using comparison tasks and asking children to compare results can help reveal the limitations of such procedures and promote more accurate measuring. However, students also need to see agreements. For example, understanding that the results of measurement and direct comparison have the same results encourages children to use measurement strategies.

Another important issue concerns the use of standard or nonstandard units of length. Many curricula or other instructional guides advise a sequence of instruction in which students compare lengths, measure with nonstandard units (e.g., paper clips), incorporate the use of manipulative standard units (e.g., inch cubes), and measure with a ruler. This approach is probably intended to help students see the need for standardization. However, the use of a variety of different length units, *before students understand the concepts, procedures, and usefulness of measurement*, may actually deter students' development. Instead, students might learn to measure correctly with standard units, and even learn to use rulers, before they can successfully use nonstandard units and understand relationships between different units of measurement. To realize that arbitrary (and especially mixed-size) units result in the same length being described by different numbers, a student must reconcile the varying lengths and numbers of arbitrary units. Emphasizing nonstandard units too early may defeat the purpose it is intended to achieve. Early use of many nonstandard units may actually interfere with students' development of basic measurement concepts required to understand the need for standard units. In contrast, using manipulative standard units, or even standard rulers, is less demanding and appears to be a more interesting and meaningful real-world activity for young students.

Thus, an instructional progression based on this finding would start by ensuring that students can perform direct comparisons. Then, children should engage in experiences that allow them to connect number to length, using manipulative units that have a standard unit of length, such as centimeter cubes. These can be labeled "length-units" with the students. Students learn to lay such physical units end-to-end and count them to measure a length. They compare the results of measuring to direct and indirect comparisons.

As they measure with these manipulative units, students discuss the concepts and skills involved (e.g., as previously discussed, not leaving space between successive length-units). As another example, students initially may not extend the unit past the endpoint of the object they are measuring. If students make procedural errors such as these, they can be asked to tell in a precise and elaborate manner *what* the problem is, *why* it leads to incorrect measurements, and *how* to fix it and measure accurately.

Measurement activities can also develop other areas of mathematics, including reasoning and logic. In one class, first graders were studying mathematics mainly through measurement, rather than counting discrete objects. They described and represented relationships among and between lengths (MP2, MP3), such as comparing two sticks and symbolizing the lengths as "$A < B$." This enabled them to reason about relationships. For example, after seeing the following statements recorded on the board, if $V > M$, then $M \neq V$, $V \neq M$, and $M < V$, one first-grader noted, "If it's an inequality, then you can write four statements. If it's equal, you can only write two"(MP8)

This indicates that with high-quality experiences (such as those described in the Grade 2 section on length), many first graders can also learn to use reasoning, connecting this to direct comparison, and to measurement performed by laying physical units end-to-end.

Area and volume: Foundations As in Kindergarten, area and volume are not instructional foci for first grade, but some everyday activities can form an experiential foundation for later instruction in these topics. For example, in later grades, understanding area requires seeing how to decompose shapes into parts and how to move and recombine the parts to make simpler shapes whose areas are already known (MP7). First graders learn the foundations of such procedures both in composing and decomposing shapes, discussed in the Geometry Progression, and in comparing areas in specific contexts. For example, paper-folding activities lend themselves not just to explorations of symmetry but also to equal-area congruent parts. Some students can compare the area of two pieces of paper by cutting and overlaying them. Such experiences provide only initial development of area concepts, but these key foundations are important for later learning.

Volume can involve liquids or solids. This leads to two ways to measure volume, illustrated by "packing" a space such as a three-dimensional array with cubic units and "filling" with iterations of a fluid unit that takes the shape of the container (called liquid volume). Many first graders initially perceive filling as having a one-dimensional unit structure. For example, students may simply "read off" the measure on a graduated cylinder. Thus, in a science or "free time" activity, students might compare the volume of two containers in at least two ways. They might pour each into a graduated cylinder to compare the measures. Or they might practice indirect comparison using transitive reasoning by using a third container to compare the volumes of the two containers. By packing unit cubes into containers into which cubes fit readily, students also can lay a foundation for later "packing" volume.

Grade 2

Measure and estimate lengths in standard units Second graders learn to measure length with a variety of tools, such as rulers, meter sticks, and measuring tapes.[2.MD.1] Although this appears to some adults to be relatively simple, there are many conceptual and procedural issues to address. For example, students may begin counting at the numeral "1" on a ruler. The numerals on a ruler may signify to students when to start counting, rather than the amount of space that has already been covered. It is vital that students learn that "one" represents the space from the beginning of the ruler to the hash mark, not the hash mark itself. Again, students may not understand that units must be of equal size. They will even measure with tools subdivided into units of different sizes and conclude that quantities with more units are larger.

To learn measurement concepts and skills, students might use both simple rulers (e.g., having only whole units such as centimeters or inches) and physical units (e.g., manipulatives that are centimeter or inch lengths). As described for Grade 1, teachers and students can call these "length-units." Initially, students lay multiple copies of the same physical unit end-to-end along the ruler. They can also progress to iterating with one physical unit (i.e., repeatedly marking off its endpoint, then moving it to the next position), even though this is more difficult physically and conceptually. To help them make the transition to this more sophisticated understanding of measurement, students might draw length unit marks along sides of geometric shapes or other lengths to see the unit lengths. As they measure with these tools, students with the help of the teacher discuss the concepts and skills involved, such as the following.

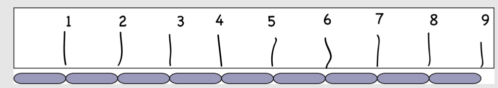
- *length-unit iteration*. E.g., not leaving space between successive length-units;
- *accumulation of distance*. Understanding that the counting "eight" when placing the last length-unit means the space covered by 8 length-units, rather then just the eighth length-unit (note the connection to cardinality[K.CC.4]);
- *alignment of zero-point*. Correct alignment of the zero-point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;
- *meaning of numerals on the ruler*. The numerals indicate the number of length units so far;
- *connecting measurement with physical units and with a ruler*. Measuring by laying physical units end-to-end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

Students also can learn accurate procedures and concepts by drawing simple unit rulers. Using copies of a single length-unit such as inch-long manipulatives, they mark off length-units on strips of paper, explicitly connecting measurement with the ruler to measurement by iterating physical units. Thus, students' first rulers should be simply ways to help count the iteration of length-units. Frequently comparing results of measuring the same object with manipulative standard units and with these rulers helps students connect their experiences and ideas. As they build and use these tools, they develop the ideas of length-unit iteration, correct alignment (with a ruler), and the zero-point concept (the idea that the zero of the ruler indicates one endpoint of a length). These are reinforced as children compare the results of measuring to compare to objects with the results of directly comparing these objects.

Using a unit to draw a ruler



Students use a standard unit (shown in below the ruler) to make rulers, helping them to understand the meaning of the markings on rulers.

Similarly, discussions might frequently focus on "*What* are you counting?" with the answer being "length-units" or "centimeters" or the like. This is especially important because counting discrete items often convinces students that the size of things counted does not matter (there could be exactly 10 toys, even if they are different sizes). In contrast, for measurement, unit size is critical, so teachers are advised to plan experiences and reflections on the use of other units and length-units in various discrete counting and measurement contexts. Given that counting discrete items often correctly teaches students that the length-unit size does not matter, so teachers are advised to plan experiences and reflections on the use of units in various discrete counting and measurement contexts. For example, a teacher might challenge students to consider a fictitious student's measurement in which he lined up three large and four small blocks and claimed a path was "seven blocks long." Students can discuss whether he is correct or not.

Second graders also learn the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a specific length or distance.[2.MD.2] For example, it will take more centimeter lengths to cover a certain distance than inch lengths because inches are the larger unit. Initially, students may not appreciate the need for identical units. Previously described work with manipulative units of standard measure (e.g., 1 inch or 1 cm), along with related use of rulers and consistent discussion, will help children learn both the concepts and procedures of linear measurement. Thus, second grade students can learn that the larger the unit, the fewer number of units in a given measurement (as was illustrated on p. 9). That is, for measurements of a given length there is an inverse relationship between the size of the unit of measure and the number of those units. This is the time that measuring and reflecting on measuring the same object with different units, both standard and nonstandard, is likely to be most productive (see the discussion of this issue in the Grade 1 section on length). Results of measuring with different nonstandard length-units can be explicitly compared. Students also can use the concept of unit to make

[2.MD.2] Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

inferences about the relative sizes of objects; for example, if object *A* is 10 regular paperclips long and object *B* is 10 jumbo paperclips long, the number of units is the same, but the units have different sizes, so the lengths of *A* and *B* are different.

Second graders also learn to combine and compare lengths using arithmetic operations. That is, they can add two lengths to obtain the length of the whole and subtract one length from another to find out the difference in lengths.[2.MD.4] For example, they can use a simple unit ruler or put a length of connecting cubes together to measure first one modeling clay "snake," then another, to find the total of their lengths. The snakes can be laid along a line, allowing students to compare the measurement of that length with the sum of the two measurements. Second graders also begin to apply the concept of length in less obvious cases, such as the width of a circle, the length and width of a rectangle, the diagonal of a quadrilateral, or the height of a pyramid. As an arithmetic example, students might measure all the sides of a table with unmarked (foot) rulers to measure how much ribbon they would need to decorate the perimeter of the table.[2.MD.5] They learn to measure two objects and subtract the smaller measurement from the larger to find how much longer one object is than the other.

Second graders can also learn to represent and solve numerical problems about length using tape or number-bond diagrams. (See p. 16 of the Operations and Algebraic Thinking Progression for discussion of when and how these diagrams are used in Grade 1.) Students might solve two-step numerical problems at different levels of sophistication (see p. 18 of the Operations and Algebraic Thinking Progression for similar two-step problems involving discrete objects). Conversely, "missing measurements" problems about length may be presented with diagrams.

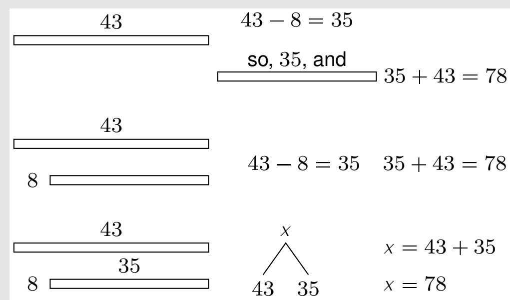
These understandings are essential in supporting work with number line diagrams.[2.MD.6] That is, to use a number line diagram to understand number and number operations, students need to understand that number line diagrams have specific conventions: the use of a single position to represent a whole number and the use of marks to indicate those positions. They need to understand that a number line diagram is like a ruler in that consecutive whole numbers are 1 unit apart, thus they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie students' successful use of number line diagrams. Students think of a number line diagram as a measurement model and use strategies relating to distance, proximity of numbers, and reference points.

After experience with measuring, second graders learn to estimate lengths.[2.MD.3] Real-world applications of length often involve estimation. Skilled estimators move fluently back and forth between written or verbal length measurements and representations of their corresponding magnitudes on a *mental ruler* (also called the "mental number line"). Although having real-world "benchmarks" is useful

[2.MD.4] Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

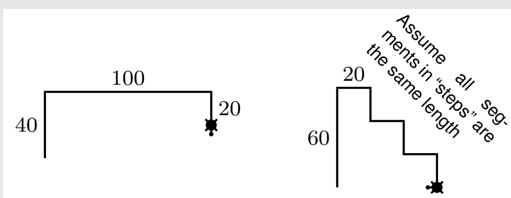
[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

Missing measurements problems



Different solution methods for "A girl had a 43 cm section of a necklace and another section that was 8 cm shorter than the first. How long the necklace would be if she combined the two sections?" 2.MD.5

Missing measurements problems



What are the missing lengths of the third and fourth sides of the rectangle?

What are the missing lengths of each step and the bottom of the stairway?

These problems might be presented in the context of turtle geometry. Students work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house (the remainder of the right side, and the bottom). They then type in Logo commands (e.g., for the rectangle, forward 40 right 90 fd 100 rt 90 fd 20 fd 20 rt 90 fd 100) to check their calculations (MP5).

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.3] Estimate lengths using units of inches, feet, centimeters, and meters.

(e.g., a meter is about the distance from the floor to the top of a doorknob), instruction should also help children build understandings of scales and concepts of measurement into their estimation competencies. Although "guess and check" experiences can be useful, research suggests explicit teaching of estimation strategies (such as iteration of a mental image of the unit or comparison with a known measurement) and prompting students to learn reference or benchmark lengths (e.g., an inch-long piece of gum, a 6-inch dollar bill), order points along a continuum, and build up mental rulers.

Length measurement should also be used in other domains of mathematics, as well as in other subjects, such as science, and connections should be made where possible. For example, a line plot scale is just a ruler, usually with a non-standard unit of length. Teachers can ask students to discuss relationships they see between rulers and line plot scales. Data using length measures might be graphed (see example on pp. 8–9 of the Measurement Data Progression). Students could also graph the results of many students measuring the same object as precisely as possible (even involving halves or fourths of a unit) and discuss what the "real" measurement of the object might be. Emphasis on students solving real measurement problems, and, in so doing, building and iterating units, as well as units of units, helps students development strong concepts and skills. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics.

Area and volume: Foundations To learn area (and, later, volume) concepts and skills meaningfully in later grades, students need to develop the ability known as *spatial structuring*. Students need to be able to see a rectangular region as decomposable into rows and columns of squares. This competence is discussed in detail in the Geometry Progression, but is mentioned here for two reasons. First, such spatial structuring precedes meaningful mathematical use of the structures, such as determining area or volume. Second, Grade 2 work in multiplication involves work with rectangular arrays,[2.G.2] and this work is an ideal context in which to simultaneously develop both arithmetical and spatial structuring foundations for later work with area.

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Grade 3

Perimeter Third graders focus on solving real-world and mathematical problems involving perimeters of polygons.[3.MD.8] A perimeter is the boundary of a two-dimensional shape. For a polygon, the length of the perimeter is the sum of the lengths of the sides. Initially, it is useful to have sides marked with unit length marks, allowing students to count the unit lengths. Later, the lengths of the sides can be labeled with numerals. As with all length tasks, students need to count the length-units and not the end-points. Next, students learn to mark off unit lengths with a ruler and label the length of each side of the polygon. For rectangles, parallelograms, and regular polygons, students can discuss and justify faster ways to find the perimeter length than just adding all of the lengths (MP3). Rectangles and parallelograms have opposite sides of equal length, so students can double the lengths of adjacent sides and add those numbers or add lengths of two adjacent sides and double that number. A regular polygon has all sides of equal length, so its perimeter length is the product of one side length and the number of sides.

Perimeter problems for rectangles and parallelograms often give only the lengths of two adjacent sides or only show numbers for these sides in a drawing of the shape. The common error is to add just those two numbers. Having students first label the lengths of the other two sides as a reminder is helpful.

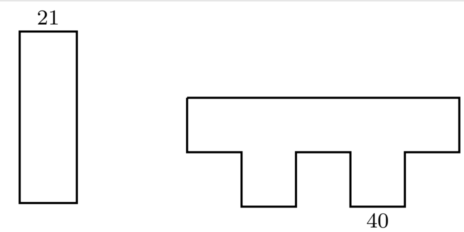
Students then find unknown side lengths in more difficult "missing measurements" problems and other types of perimeter problems.[3.MD.8]

Children learn to subdivide length-units. Making one's own ruler and marking halves and other partitions of the unit may be helpful in this regard. For example, children could fold a unit in halves, mark the fold as a half, and then continue to do so, to build fourths and eighths, discussing issues that arise. Such activities relate to fractions on the number line.[3.NF.2] Labeling all of the fractions can help students understand rulers marked with halves and fourths but not labeled with these fractions. Students also measure lengths using rulers marked with halves and fourths of an inch.[3.MD.4] They show these data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters (see the Measurement Data Progression, p. 10).

Understand concepts of area and relate area to multiplication and to addition Third graders focus on learning area. Students learn formulas to compute area, with those formulas based on, and summarizing, a firm conceptual foundation about what area is. Students need to learn to conceptualize area as the amount of two-dimensional space in a bounded region and to measure it by choosing a unit of area, often a square. A two-dimensional geometric figure that is covered by a certain number of squares without gaps

[3.MD.8] Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Missing measurements and other perimeter problems



The perimeter of this rectangle is 168 length units. What are the lengths of the three unlabeled sides?

Assume all short segments are the same length and all angles are right

Compare these problems with the "missing measurements" problems of Grade 2.

Another type of perimeter problem is to draw a robot on squared grid paper that meets specific criteria. All the robot's body parts must be rectangles. The perimeter of the head might be 36 length-units, the body, 72; each arm, 24; and each leg, 72. Students are asked to provide a convincing argument that their robots meet these criteria (MP3). Next, students are asked to figure out the area of each of their body parts (in square units). These are discussed, with students led to reflect on the different areas that may be produced with rectangles of the same perimeter. These types of problems can be also presented as turtle geometry problems. Students create the commands on paper and then give their commands to the Logo turtle to check their calculations. For turtle length units, the perimeter of the head might be 300 length-units, the body, 600; each arm, 400; and each leg, 640.

[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

[3.MD.4] Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

or overlaps can be said to have an area of that number of square units.[3.MD.5]

Activities such as those in the Geometry Progression teach students to compose and decompose geometric regions. To begin an explicit focus on area, teachers might then ask students which of three rectangles covers the most area. Students may first solve the problem with decomposition (cutting and/or folding) and re-composition, and eventually analyses with area-units, by covering each with unit squares (tiles).[3.MD.5, 3.MD.6] Discussions should clearly distinguish the attribute of area from other attributes, notably length.

Students might then find the areas of other rectangles. As previously stated, students can be taught to multiply length measurements to find the area of a rectangular region. But, in order that they make sense of these quantities (MP2), they first learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows.[3.MD.7a] This relies on the development of spatial structuring.[MP7] To build from spatial structuring to understanding the number of area-units as the product of number of units in a row and number of rows, students might draw rectangular arrays of squares and learn to determine the number of squares in each row with increasingly sophisticated strategies, such as skip-counting the number in each row and eventually multiplying the number in each row by the number of rows (MP8). They learn to partition a rectangle into identical squares by anticipating the final structure and forming the array by drawing line segments to form rows and columns. They use skip counting and multiplication to determine the number of squares in the array.

Many activities that involve seeing and making arrays of squares to form a rectangle might be needed to build robust conceptions of a rectangular area structured into squares. One such activity is illustrated in the margin. In this progression, less sophisticated activities of this sort were suggested for earlier grades so that Grade 3 students begin with some experience.

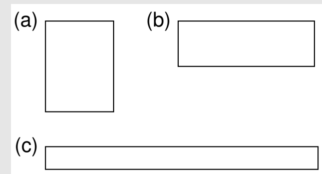
Students learn to understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior (MP3).[3.MD.7a] For example, students might explain that one length tells how many unit squares in a row and the other length tells how many rows there are.

Students might then solve numerous problems that involve rectangles of different dimensions (e.g., designing a house with rooms that fit specific area criteria) to practice using multiplication to compute areas.[3.MD.7b] The areas involved should not all be rectangular, but decomposable into rectangles (e.g., an "L-shaped" room).[3.MD.7d]

Students also might solve problems such as finding all the rectangular regions with whole-number side lengths that have an area of 12 area-units, doing this later for larger rectangles (e.g., enclosing 24, 48, or 72 area-units), making sketches rather than drawing each

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

Which rectangle covers the most area?



These rectangles are formed from unit squares (tiles students have used) although students are not informed of this or the rectangle's dimensions: (a) 4 by 3, (b) 2 by 6, and (c) 1 row of 12. Activity from Lehrer, et al., 1998, "Developing understanding of geometry and space in the primary grades," in R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*, Lawrence Erlbaum Associates.

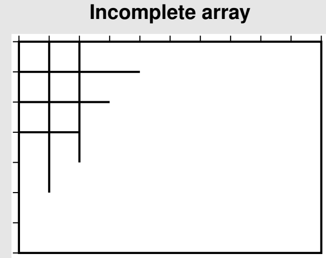
3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

MP7 See the Geometry Progression

Incomplete array



To determine the area of this rectangular region, students might be encouraged to construct a row, corresponding to the indicated positions, then repeating that row to fill the region. Cutouts of strips of rows can help the needed spatial structuring and reduce the time needed to show a rectangle as rows or columns of squares. Drawing all of the squares can also be helpful, but it is slow for larger rectangles. Drawing the unit lengths on the opposite sides can help students see that joining opposite unit end-points will create the needed unit square grid.

3.MD.7b Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

3.MD.7d Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

square. They learn to justify their belief they have found all possible solutions (MP3).

Similarly using concrete objects or drawings, and their competence with composition and decomposition of shapes, spatial structuring, and addition of area measurements, students learn to investigate arithmetic properties using area models. For example, they learn to rotate rectangular arrays physically and mentally, understanding that their areas are preserved under rotation, and thus, for example, $4 \times 7 = 7 \times 4$, illustrating the commutative property of multiplication.[3.MD.7c] They also learn to understand and explain that the area of a rectangular region of, for example, 12 length-units by 5 length-units can be found either by multiplying 12×5, or by adding two products, e.g., 10×5 and 2×5, illustrating the distributive property.

3.MD.7c Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures With strong and distinct concepts of both perimeter and area established, students can work on problems to differentiate their measures. For example, they can find and sketch rectangles with the same perimeter and different areas or with the same area and different perimeters and justify their claims (MP3).[3.MD.8] Differentiating perimeter from area is facilitated by having students draw congruent rectangles and measure, mark off, and label the unit lengths all around the perimeter on one rectangle, then do the same on the other rectangle but also draw the square units. This enables students to see the units involved in length and area and find patterns in finding the lengths and areas of non-square and square rectangles (MP7). Students can continue to describe and show the units involved in perimeter and area after they no longer need these .

3.MD.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Problem solving involving measurement and estimation of intervals of time, liquid volumes, and masses of objects Students in Grade 3 learn to solve a variety of problems involving measurement and such attributes as length and area, liquid volume, mass, and time.[3.MD.1, 3.MD.2] Many such problems support the Grade 3 emphasis on multiplication (see Table 1) and the mathematical practices of making sense of problems (MP1) and representing them with equations, drawings, or diagrams (MP4). Such work will involve units of mass such as the kilogram.

3.MD.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

3.MD.2 Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[2] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[3]

Table 1: Multiplication and division situations for measurement

	Unknown Product $A \times B = \square$	Group Size Unknown $A \times \square = C$ and $C \div A = \square$	Number of Groups Unknown $\square \times B = C$ and $C \div B = \square$
Grouped Objects (Units of Units)	You need A lengths of string, each B inches long. How much string will you need altogether?	You have C inches of string, which you will cut into A equal pieces. How long will each piece of string be?	You have C inches of string, which you will cut into pieces that are B inches long. How many pieces of string will you have?
Arrays of Objects (Spatial Structuring)	What is the area of a A cm by B cm rectangle?	A rectangle has area C square centimeters. If one side is A cm long, how long is a side next to it?	A rectangle has area C square centimeters. If one side is B cm long, how long is a side next to it?
Compare	A rubber band is B cm long. How long will the rubber band be when it is stretched to be A times as long?	A rubber band is stretched to be C cm long and that is A times as long as it was at first. How long was the rubber band at first?	A rubber band was B cm long at first. Now it is stretched to be C cm long. How many times as long is the rubber band now as it was at first?

Adapted from box 2-4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33. Note that Grade 3 work does not include Compare problems with “times as much,” see the Operations and Algebraic Thinking Progression, Table 3, also p. 29.

A few words on volume are relevant. Compared to the work in area, volume introduces more complexity, not only in adding a third dimension and thus presenting a significant challenge to students' spatial structuring, but also in the materials whose volumes are measured. These materials may be solid or fluid, so their volumes are generally measured with one of two methods, e.g., “packing” a right rectangular prism with cubic units or “filling” a shape such as a right circular cylinder. Liquid measurement, for many third graders, may be limited to a one-dimensional unit structure (i.e., simple iterative counting of height that is not processed as three-dimensional). Thus, third graders can learn to measure with liquid volume and to solve problems requiring the use of the four arithmetic operations, when liquid volumes are given in the same units throughout each problem. Because liquid measurement can be represented with one-dimensional scales, problems may be presented with drawings or diagrams, such as measurements on a beaker with a measurement scale in milliliters.

Grade 4

In Grade 4, students build on competencies in measurement and in building and relating units and units of units that they have developed in number, geometry, and geometric measurement.

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit Fourth graders learn the relative sizes of measurement units within a system of measurement[4.MD.1] including:

> *length*: meter (m), kilometer (km), centimeter (cm), millimeter (mm); volume: liter (l), milliliter (ml, 1 cubic centimeter of water; a liter, then, is 1000 ml);
>
> *mass*: gram (g, about the weight of a cc of water), kilogram (kg); time: hour (hr), minute (min), second (sec).

For example, students develop benchmarks and mental images about a meter (e.g., about the height of a tall chair) and a kilometer (e.g., the length of 10 football fields including the end zones, or the distance a person might walk in about 12 minutes), and they also understand that "kilo" means a thousand, so 3000 m is equivalent to 3 km.

Expressing larger measurements in smaller units within the metric system is an opportunity to reinforce notions of place value. There are prefixes for multiples of the basic unit (meter or gram), although only a few (kilo-, centi-, and milli-) are in common use. Tables such as the one in the margin indicate the meanings of the prefixes by showing them in terms of the basic unit (in this case, meters). Such tables are an opportunity to develop or reinforce place value concepts and skills in measurement activities.

Relating units within the metric system is another opportunity to think about place value. For example, students might make a table that shows measurements of the same lengths in centimeters and meters.

Relating units within the traditional system provides an opportunity to engage in mathematical practices, especially "look for and make use of structure" (MP7) and "look for and express regularity in repeated reasoning" (MP8). For example, students might make a table that shows measurements of the same lengths in feet and inches.

Students also combine competencies from different domains as they solve measurement problems using all four arithmetic operations, addition, subtraction, multiplication, and division (see examples in Table 1).[4.MD.2] For example, "How many liters of juice does the class need to have at least 35 cups if each cup takes 225 ml?" Students may use tape or number line diagrams for solving such problems (MP1).

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

Super- or subordinate unit	Length in terms of basic unit
kilometer	10^3 or 1000 meters
hectometer	10^2 or 100 meters
decameter	10^1 or 10 meters
meter	1 meter
decimeter	10^{-1} or $\frac{1}{10}$ meters
centimeter	10^{-2} or $\frac{1}{100}$ meters
millimeter	10^{-3} or $\frac{1}{1000}$ meters

Note the similarity to the structure of base-ten units and U.S. currency (see illustrations on p. 12 of the Number and Operations in Base Ten Progression).

Centimeter and meter equivalences

cm	m
100	1
200	2
300	3
500	
1000	

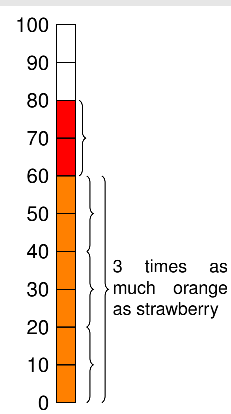
Foot and inch equivalences

feet	inches
0	0
1	12
2	24
3	

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Using tape diagrams to solve word problems

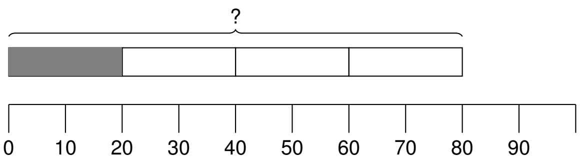
Lisa put two flavors of soda in a glass. There were 80 ml of soda in all. She put three times as much orange drink as strawberry. How many ml of orange did she put in?



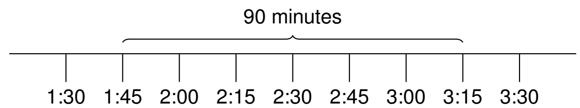
In this diagram, quantities are represented on a measurement scale.

Using number line diagrams to solve word problems

Juan spent 1/4 of his money on a game. The game cost $20. How much money did he have at first?



What time does Marla have to leave to be at her friend's house by a quarter after 3 if the trip takes 90 minutes?



Using a number line diagram to represent time is easier if students think of digital clocks rather than round clocks. In the latter case, placing the numbers on the number line involves considering movements of the hour and minute hands.

Students learn to consider perimeter and area of rectangles, begun in Grade 3, more abstractly (MP2). Based on work in previous grades with multiplication, spatially structuring arrays, and area, they abstract the formula for the area of a rectangle $A = l \times w$.•

• The formula is a generalization of the understanding, that, given a unit of length, a rectangle whose sides have length w units and l units, can be partitioned into w rows of unit squares with l squares in each row. The product $l \times w$ gives the number of unit squares in the partition, thus the area measurement is $l \times w$ square units. These square units are derived from the length unit.

Students generate and discuss advantages and disadvantages of various formulas for the perimeter length of a rectangle that is l units by w units.• Giving verbal summaries of these formulas is also helpful. For example, a verbal summary of the basic formula, $A = l + w + l + w$, is "add the lengths of all four sides." Specific numerical instances of other formulas or mental calculations for the perimeter of a rectangle can be seen as examples of the properties of operations, e.g., $2l + 2w = 2(l + w)$ illustrates the distributive property.

• For example, $P = 2l + 2w$ has two multiplications and one addition, but $P = 2(l + w)$, which has one addition and one multiplication, involves fewer calculations. The latter formula is also useful when generating all possible rectangles with a given perimeter. The length and width vary across all possible pairs whose sum is half of the perimeter (e.g., for a perimeter of 20, the length and width are all of the pairs of numbers with sum 10).

Perimeter problems often give only one length and one width, thus remembering the basic formula can help to prevent the usual error of only adding one length and one width. The formula $P = 2(l + w)$ emphasizes the step of multiplying the total of the given lengths by 2. Students can make a transition from showing all length units along the sides of a rectangle or all area units within (as in Grade 3, p. 18) by drawing a rectangle showing just parts of these as a reminder of which kind of unit is being used. Writing all of the lengths around a rectangle can also be useful. Discussions of formulas such as $P = 2l + 2w$, can note that unlike area formulas, perimeter formulas combine length measurements to yield a length measurement.

Such abstraction and use of formulas underscores the importance of distinguishing between area and perimeter in Grade 3[3.MD.8] and maintaining the distinction in Grade 4 and later grades, where rectangle perimeter and area problems may get more complex and problem solving can benefit from knowing or being able to rapidly remind oneself of how to find an area or perimeter. By repeatedly reasoning about how to calculate areas and perimeters of rectangles, students can come to see area and perimeter formulas as summaries of all such calculations (MP8).

[3.MD.8] Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Students learn to apply these understandings and formulas to the solution of real-world and mathematical problems.[4.MD.3] For example, they might be asked, "A rectangular garden has as an area of 80 square feet. It is 5 feet wide. How long is the garden?" Here, specifying the area and the width, creates an unknown factor problem (see Table 1). Similarly, students could solve perimeter problems that give the perimeter and the length of one side and ask the length of the adjacent side. Students could be challenged to solve multistep problems such as the following. "A plan for a house includes rectangular room with an area of 60 square meters and a perimeter of 32 meters. What are the length and the width of the room?"

4.MD.3 Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

In Grade 4 and beyond, the mental visual images for perimeter and area from Grade 3 can support students in problem solving with these concepts. When engaging in the mathematical practice of reasoning abstractly and quantitatively (MP2) in work with area and perimeter, students think of the situation and perhaps make a drawing. Then they recreate the "formula" with specific numbers and one unknown number as a situation equation for this particular numerical situation.• "Apply the formula" does not mean write down a memorized formula and put in known values because at Grade 4 students do not evaluate expressions (they begin this type of work in Grade 6). In Grade 4, working with perimeter and area of rectangles is still grounded in specific visualizations and numbers. These numbers can now be any of the numbers used in Grade 4 (for addition and subtraction for perimeter and for multiplication and division for area).[4.NBT.4, 4.NF.3d, 4.OA.4] By repeatedly reasoning about constructing situation equations for perimeter and area involving specific numbers and an unknown number, students will build a foundation for applying area, perimeter, and other formulas by substituting specific values for the variables in later grades.

• "Situation equation" refers to the idea that the student constructs an equation as a representation of a situation rather than identifying the situation as an example of a familiar equation.

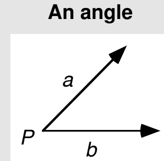
4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

4.NF.3d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

4.OA.4 Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Understand concepts of angle and measure angles Angle measure is a "turning point" in the study of geometry. Students often find angles and angle measure to be difficult concepts to learn, but that learning allows them to engage in interesting and important mathematics. An *angle* is the union of two rays, a and b, with the same initial point P. The rays can be made to coincide by rotating one to the other about P; this rotation determines the size of the angle between a and b. The rays are sometimes called the *sides* of the angles.

An angle



P is called the *vertex* of the angle and the rays a and b are called the *arms*.

Another way of saying this is that each ray determines a direction and the angle size measures the change from one direction to the other. (This illustrates how angle measure is related to the concepts of parallel and perpendicular lines in Grade 4 geometry.) A clockwise rotation is considered positive in surveying or turtle geometry; but a counterclockwise rotation is considered positive in Euclidean geometry. Angles are measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and degrees are the unit used to measure angles in elementary school. A full rotation is thus 360°.

Two angles are called *complementary* if their measurements have the sum of 90°. Two angles are called *supplementary* if their measurements have the sum of 180°. Two angles with the same vertex that overlap only at a boundary (i.e., share a side) are called *adjacent angles*.

Like length, area, and volume, angle measure is additive: The sum of the measurements of *adjacent angles* is the measurement of the angle formed by their union. This leads to other important properties. If a right angle is decomposed into two adjacent angles, the sum is 90°, thus they are complementary. Two adjacent angles that compose a "straight angle" of 180° must be supplementary. In some situations (see margin), such properties allow logical progressions of statements (MP3).

As with all measureable attributes, students must first recognize the attribute of angle measure, and distinguish it from other attributes. This may not appear too difficult, as the measure of angles and rotations appears to knowledgeable adults as quite different than attributes such as length and area. However, the unique nature of angle size leads many students to initially confuse angle measure with other, more familiar, attributes. Even in contexts designed to evoke a dynamic image of turning, such as hinges or doors, many students use the length between the endpoints, thus teachers find it useful to repeatedly discuss such cognitive "traps."

As with other concepts (e.g., see the Geometry Progression), students need varied examples and explicit discussions to avoid learning limited ideas about measuring angles (e.g., misconceptions that a right angle is an angle that points to the right, or two right angles represented with different orientations are not equal in measure). If examples and tasks are not varied, students can develop incomplete and inaccurate notions. For example, some come to associate all slanted lines with 45° measures and horizontal and vertical lines with measures of 90°. Others believe angles can be "read off" a protractor in "standard" position, that is, a base is horizontal, even if neither arm of the angle is horizontal. Measuring and then sketching many angles with no horizontal or vertical arms,[4.MD.6] perhaps initially using circular 360° protractors, can help students avoid such limited conceptions.

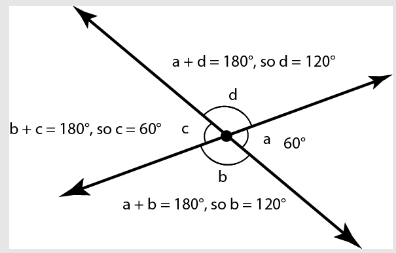
As with length, area, and volume, children need to understand equal partitioning and unit iteration to understand angle and turn measure. Whether defined as more statically as the measure of the figure formed by the intersection of two rays or as turning, having a given angle measure involves a relationship between components of plane figures and therefore is a *property* (see the Overview in the Geometry Progression).[4.G.2]

Given the complexity of angles and angle measure, it is unsur-

An angle

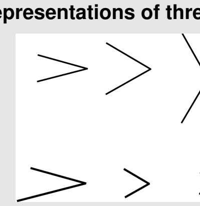
name	measurement
right angle	90°
straight angle	180°
acute angle	between 0 and 90°
obtuse angle	between 90° and 180°
reflex angle	between 180° and 360°

Angles created by the intersection of two lines



When two lines intersect, they form four angles. If the measurement of one is known (e.g., angle a is 60°), the measurement of the other three can be determined.

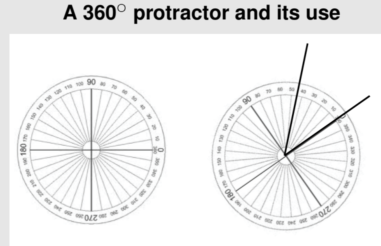
Two representations of three angles



Initially, some students may correctly compare angle sizes only if all the line segments are the same length (as shown in the top row). If the lengths of the line segments are different (as shown in the bottom row), these students base their judgments on the lengths of the segments, the distances between their endpoints, or even the area of the triangles determined by the drawn arms. They believe that the angles in the bottom row decrease in size from left to right, although they have, respectively, the same angle measurements as those in the top row.

4.MD.6 Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

A 360° protractor and its use



The figure on the right shows a protractor being used to measure a 45° angle. The protractor is placed so that one side of the angle lies on the line corresponding to 0° on the protractor and the other side of the angle is located by a clockwise rotation from that line.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

prising that students in the early and elementary grades often form separate concepts of angles as figures and turns, and may have separate notions for different turn contexts (e.g., unlimited rotation as a fan vs. a hinge) and for various "bends."

However, students can develop more accurate and useful angle and angle measure concepts if presented with angles in a variety of situations. They learn to find the common features of superficially different situations such as turns in navigation, slopes, bends, corners, and openings. With guidance, they learn to represent an angle in any of these contexts as two rays, even when both rays are not explicitly represented in the context; for example, the horizontal or vertical in situations that involve slope (e.g., roads or ramps), or the angle determined by looking up from the horizon to a tree- or mountain-top. Eventually they abstract the common attributes of the situations as angles (which are represented with rays and a vertex, MP4) and angle measurements (MP2). To accomplish the latter, students integrate turns, and a general, dynamic understanding of angle measure-as-rotation, into their understandings of angles-as-objects. Computer manipulatives and tools can help children bring such a dynamic concept of angle measure to an explicit level of awareness. For example, dynamic geometry environments can provide multiple linked representations, such as a screen drawing that students can "drag" which is connected to a numerical representation of angle size. Games based on similar notions are particularly effective when students manipulate not the arms of the angle itself, but a representation of rotation (a small circular diagram with radii that, when manipulated, change the size of the target angle turned).

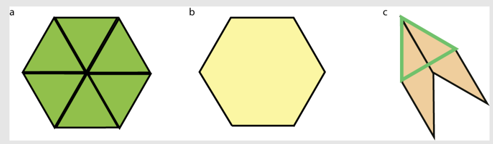
Students with an accurate conception of angle can recognize that angle measure is *additive*.[4.MD.7] As with length, area, and volume, when an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Students can then solve interesting and challenging addition and subtraction problems to find the measurements of unknown angles on a diagram in real world and mathematical problems. For example, they can find the measurements of angles formed a pair of intersecting lines, as illustrated above, or given a diagram showing the measurement of one angle, find the measurement of its complement. They can use a protractor to check, not to check their reasoning, but to ensure that they develop full understanding of the mathematics and mental images for important benchmark angles (e.g., 30°, 45°, 60°, and 90°).

Such reasoning can be challenged with many situations as illustrated in the margin.

Similar activities can be done with drawings of shapes using right angles and half of a right angle to develop the important benchmarks of 90° and 45°.

Missing measures can also be done in the turtle geometry context, building on the previous work. Note that unguided use of Logo's turtle geometry does not necessary develop strong angle

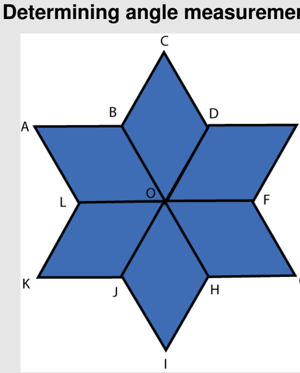
Determining angles in pattern blocks



Students might determine all the angles in the common "pattern block" shape set based on equilateral triangles. Placing six equilateral triangles so that they share a common vertex (as shown in part a), students can figure out that because the sum of the angles at this vertex is 360°, each angle which shares this vertex must have measure 60°. Because they are congruent, all the angles of the equilateral triangles must have measure 60° (again, to ensure they develop a firm foundation, students can verify these for themselves with a protractor). Because each angle of the regular hexagon (part b) is composed of two angles from equilateral triangles, the hexagon's angles each measure 120°. Similarly, in a pattern block set, two of the smaller angles from tan rhombi compose an equilateral triangle's angle, so each of the smaller rhombus angles has measure 30°.

4.MD.7 Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Determining angle measurements



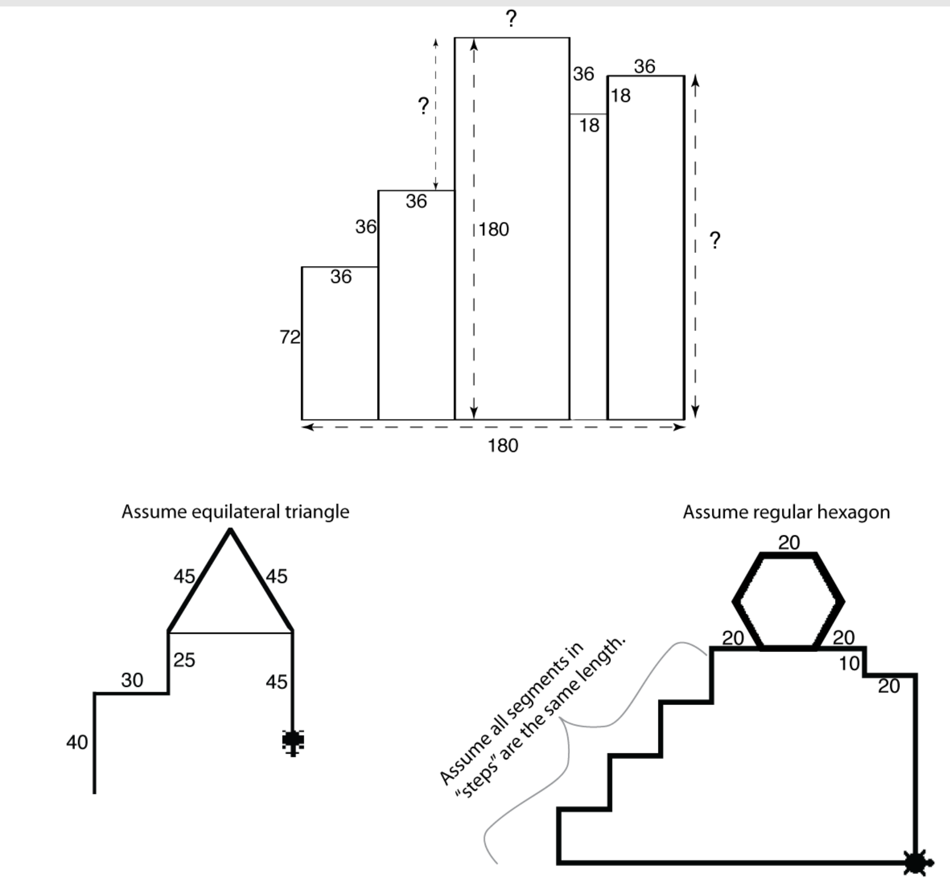
Students might be asked to determine the measurements of the following angles:

$\angle BOD$
$\angle BOF$
$\angle ODE$
$\angle CDE$
$\angle CDJ$
$\angle BHG$

concepts. However, if teachers emphasize mathematical tasks and, within those tasks, the difference between the angle of rotation the turtle makes (in a polygon, the external angle) and the angle formed (internal angle) and integrates the two, students can develop accurate and comprehensive understandings of angle measure. For example, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such questions help to connect what are often initially isolated ideas about angle conceptions.

These understandings support students in finding all the missing length and angle measures in situations such as the examples in the margin (compare to the missing measures problems Grade 2 and Grade 3).

Missing measures: Length (top) and length and angle (turn)



Students are asked to determine the missing lengths. They might first work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house, then type in Logo commands to verify their reasoning and calculations.

Grade 5

Convert like measurement units within a given measurement system In Grade 5, students extend their abilities from Grade 4 to express measurements in larger or smaller units within a measurement system.[4.MD.1, 5.MD.1] This is an excellent opportunity to reinforce notions of place value for whole numbers and decimals, and connection between fractions and decimals (e.g., $2\frac{1}{2}$ meters can be expressed as 2.5 meters or 250 centimeters). For example, building on the table from Grade 4, Grade 5 students might complete a table of equivalent measurements in feet and inches.

Grade 5 students also learn and use such conversions in solving multi-step, real world problems (see example in the margin).

Understand concepts of volume and relate volume to multiplication and to addition The major emphasis for measurement in Grade 5 is volume. Volume not only introduces a third dimension and thus a significant challenge to students' spatial structuring, but also complexity in the nature of the materials measured. That is, solid units are "packed," such as cubes in a three-dimensional array, whereas a liquid "fills" three-dimensional space, taking the shape of the container. As noted earlier (see Overview, also Grades 1 and 3), the unit structure for liquid measurement may be psychologically one-dimensional for some students.

"Packing" volume is more difficult than iterating a unit to measure length and measuring area by tiling. Students learn about a unit of volume, such as a cube with a side length of 1 unit, called a unit cube.[5.MD.3] They pack cubes (without gaps) into right rectangular prisms and count the cubes to determine the volume or build right rectangular prisms from cubes and see the layers as they build.[5.MD.4] They can use the results to compare the volume of right rectangular prisms that have different dimensions. Such experiences enable students to extend their spatial structuring from two to three dimensions (see the Geometry Progression). That is, they learn to both mentally decompose and recompose a right rectangular prism built from cubes into layers, each of which is composed of rows and columns. That is, given the prism, they have to be able to decompose it, understanding that it can be partitioned into layers, and each layer partitioned into rows, and each row into cubes. They also have to be able to compose such as structure, multiplicatively, back into higher units. That is, they eventually learn to conceptualize a layer as a unit that itself is composed of units of units—rows, each row composed of individual cubes—and they iterate that structure. Thus, they might predict the number of cubes that will be needed to fill a box given the net of the box.

Another complexity of volume is the connection between "packing" and "filling." Often, for example, students will respond that a box can be filled with 24 centimeter cubes, or build a structure of 24 cubes, and still think of the 24 as individual, often discrete, not

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

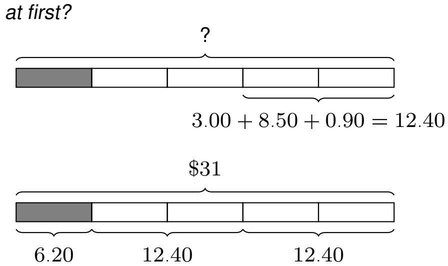
5.MD.1 Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Feet	Inches
0	0
	1
	2
	3

In Grade 6, this table can be discussed in terms of ratios and proportional relationships (see the Ratio and Proportion Progression). In Grade 5, however, the main focus is on arriving at the measurements that generate the table.

Multi-step problem with unit conversion

Kumi spent a fifth of her money on lunch. She then spent half of what remained. She bought a card game for $3, a book for $8.50, and candy for 90 cents. How much money did she have at first?

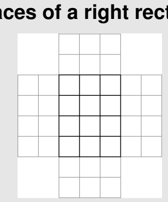


Students can use tape diagrams to represent problems that involve conversion of units, drawing diagrams of important features and relationships (MP1).

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

Net for five faces of a right rectangular prism



Students are given a net and asked to predict the number of cubes required to fill the container formed by the net. In such tasks, students may initially count single cubes or repeatedly add the number of cubes in a row to determine the number in each layer, and repeatedly add the number in each layer to find the total number of unit cubes. In folding the net to make the shape, students can see how the side rectangles fit together and determine the number of layers.

necessarily *units of volume*. They may, for example, not respond confidently and correctly when asked to fill a graduated cylinder marked in cubic centimeters with the amount of liquid that would fill the box. That is, they have not yet connected their ideas about filling volume with those concerning packing volume. Students learn to move between these conceptions, e.g., using the same container, both filling (from a graduated cylinder marked in ml or cc) and packing (with cubes that are each 1 cm^3). Comparing and discussing the volume-units and what they represent can help students learn a general, complete, and interconnected conceptualization of volume as filling three-dimensional space.

Students then learn to determine the volumes of several right rectangular prisms, using cubic centimeters, cubic inches, and cubic feet. With guidance, they learn to increasingly apply multiplicative reasoning to determine volumes, looking for and making use of structure (MP7). That is, they understand that multiplying the length times the width of a right rectangular prism can be viewed as determining how many cubes would be in each layer if the prism were packed with or built up from unit cubes.[5.MD.5a] They also learn that the height of the prism tells how many layers would fit in the prism. That is, they understand that volume is a derived attribute that, once a length unit is specified, can be computed as the product of three length measurements or as the product of one area and one length measurement.

Then, students can learn the formulas $V = l \times w \times h$ and $V = B \times h$ for right rectangular prisms as efficient methods for computing volume, maintaining the connection between these methods and their previous work with computing the number of unit cubes that pack a right rectangular prism.[5.MD.5b] They use these competencies to find the volumes of right rectangular prisms with edges whose lengths are whole numbers and solve real-world and mathematical problems involving such prisms.

Students also recognize that volume is additive (see Overview) and they find the total volume of solid figures composed of two right rectangular prisms.[5.MD.5c] For example, students might design a science station for the ocean floor that is composed of several rooms that are right rectangular prisms and that meet a set criterion specifying the total volume of the station. They draw their station (e.g., using an isometric grid, MP7) and justify how their design meets the criterion (MP1).

[5.MD.5a] Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

[5.MD.5b] Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

[5.MD.5c] Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Where the Geometric Measurement Progression is heading

Connection to Geometry In Grade 6, students build on their understanding of length, area, and volume measurement, learning to how to compute areas of right triangles and other special figures and volumes of right rectangular prisms that do not have measurements given in whole numbers. To do this, they use dissection arguments. These rely on the understanding that area and volume measures are additive, together with decomposition of plane and solid shapes (see the K–5 Geometry Progression) into shapes whose measurements students already know how to compute (MP1, MP7). In Grade 7, they use their understanding of length and area in learning and using formulas for the circumference and area of circles. In Grade 8, they use their understanding of volume in learning and using formulas for the volumes of cones, cylinders, and spheres. In high school, students learn formulas for volumes of pyramids and revisit the formulas from Grades 7 and 8, explaining them with dissection arguments, Cavalieri's principle, and informal limit arguments.

Connection to the Number System In Grade 6, understanding of length-units and spatial structuring comes into play as students learn to plot points in the coordinate plane.

Connection to Ratio and Proportion Students use their knowledge of measurement and units of measurement in Grades 6–8, coming to see conversions between two units of measurement as describing proportional relationships.

K–3, Categorical Data
2–5, Measurement Data

Progressions for the Common Core State Standards in Mathematics (draft)

20 June 2011

K–3, Categorical Data; Grades 2–5, Measurement Data*

Overview

As students work with data in Grades K–5, they build foundations for their study of statistics and probability in Grades 6 and beyond, and they strengthen and apply what they are learning in arithmetic. Kindergarten work with data uses counting and order relations. First- and second-graders solve addition and subtraction problems in a data context. In Grades 3–5, work with data is closely related to the number line, fraction concepts, fraction arithmetic, and solving problems that involve the four operations. See Table 1 for these and other notable connections between arithmetic and data work in Grades K–5.

As shown in Table 1, the K–5 data standards run along two paths. One path deals with *categorical data* and focuses on bar graphs as a way to represent and analyze such data. Categorical data comes from sorting objects into categories—for example, sorting a jumble of alphabet blocks to form two stacks, a stack for vowels and a stack for consonants. In this case there are two categories (Vowels and Consonants). Students' work with categorical data in early grades will support their later work with bivariate categorical data and two-way tables in eighth grade (this is discussed further at the end of the Categorical Data Progression).

The other path deals with *measurement data*. As the name suggests, measurement data comes from taking measurements. For example, if every child in a class measures the length of his or her hand to the nearest centimeter, then a set of measurement data is obtained. Other ways to generate measurement data might include measuring liquid volumes with graduated cylinders or measuring room temperatures with a thermometer. In each case, the Standards call for students to represent measurement data with a *line plot*.

*These progressions concern Measurement and Data standards related to data. Other MD standards are discussed in the Geometric Measurement Progression.

This is a type of display that positions the data along the appropriate scale, drawn as a number line diagram. These plots have two names in common use, "dot plot" (because each observation is represented as a dot) and "line plot" (because each observation is represented above a number line diagram).

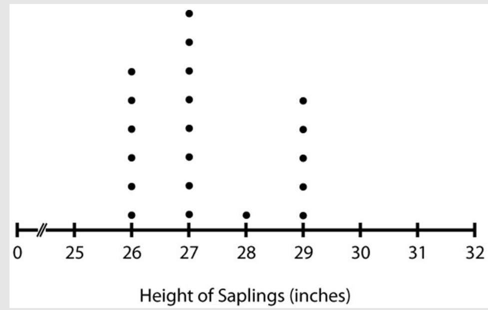
The number line diagram in a line plot corresponds to the scale on the measurement tool used to generate the data. In a context involving measurement of liquid volumes, the scale on a line plot could correspond to the scale etched on a graduated cylinder. In a context involving measurement of temperature, one might imagine a picture in which the scale on the line plot corresponds to the scale printed on a thermometer. In the last two cases, the correspondence may be more obvious when the scale on the line plot is drawn vertically.

Students should understand that the numbers on the scale of a line plot indicate the total number of measurement units from the zero of the scale.

Students need to choose appropriate representations (MP5), labeling axes to clarify the correspondence with the quantities in the situation and specifying units of measure (MP6). Measuring and recording data require attention to precision (MP6). Students should be supported as they learn to construct picture graphs, bar graphs, and line plots. Grid paper should be used for assignments as well as assessments. This may help to minimize errors arising from the need to track across a graph visually to identify values. Also, a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on. It might also help if students write relevant numbers on graphs during problem solving.

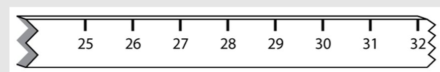
In students' work with data, context is important. As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."• In keeping with this perspective, students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the data they represent (MP2).

Example of a line plot



Note that the break in the scale between 0 and 25 indicates that marks between 0 and 25 are not shown.

Example of a scale on a measurement tool



• The *Guidelines for Assessment and Instruction in Statistics Education Report* was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

Table 1: Some notable connections to K–5 data work

Grade	Standard	Notable Connections
	Categorical data	
K	K.MD.3. Classify objects into given categories, count the number of objects in each category and sort[1] the categories by count. *Limit category counts to be less than or equal to 10.*	• K.CC. Counting to tell the number of objects • K.CC. Comparing numbers

1	1.MD.4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.	• 1.OA. Problems involving addition and subtraction
		  ◦ put-together, take-apart, compare
		  ◦ problems that call for addition of three whole numbers

2	2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.	• 2.OA. Problems involving addition and subtraction ◦ put-together, take-apart, compare
3	3.MD.3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*	• 3.OA.3. Problems involving multiplication • 3.OA.8 Two-step problems using the four operations • 3.G.1 Categories of shapes
	Measurement data	
2	2.MD.9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.	• 1.MD.2. Length measurement • 2.MD.6. Number line
3	3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units–whole numbers, halves, or quarters.	• 3.NF.2. Fractions on a number line
4	4.MD.4. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*	• 4.NF.3,4. Problems involving fraction arithmetic
5	5.MD.2. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*	• 5.NF.1,2,4,6,7. Problems involving fraction arithmetic

[1] Here, "sort the categories" means "order the categories," i.e., show the categories in order according to their respective counts.

5

Categorical Data

Kindergarten

Students in Kindergarten classify objects into categories, initially specified by the teacher and perhaps eventually elicited from students. For example, in a science context, the teacher might ask students in the class to sort pictures of various organisms into two piles: organisms with wings and those without wings. Students can then count the number of specimens in each pile.[K.CC.5] Students can use these category counts and their understanding of cardinality to say whether there are more specimens with wings or without wings.[K.CC.6,K.CC.7]

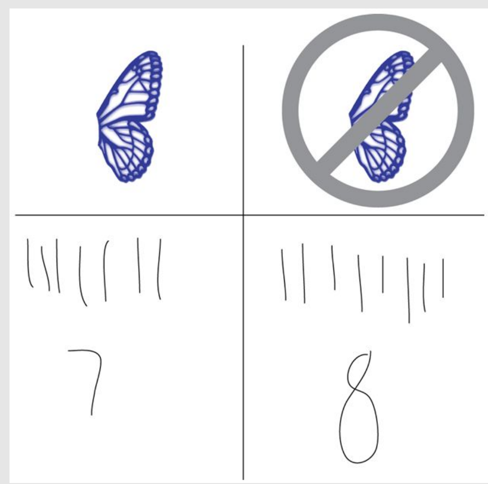
A single group of specimens might be classified in different ways, depending on which attribute has been identified as the attribute of interest. For example, some specimens might be insects, while others are not insects. Some specimens might live on land, while others live in water.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

Sorting categorical data



The marks represent individual data points. The two category counts, 7 and 8, are a numerical summary *of the data.*

Grade 1

Students in Grade 1 begin to organize and represent categorical data. For example, if a collection of specimens is sorted into two piles based on which specimens have wings and which do not, students might represent the two piles of specimens on a piece of paper, by making a group of marks for each pile, as shown below (the marks could also be circles, for example). The groups of marks should be clearly labeled to reflect the attribute in question.

The work shown in the figure is the result of an intricate process. At first, we have before us a jumble of specimens with many attributes. Then there is a narrowing of attention to a single attribute (wings or not). Then the objects might be arranged into piles. The arranging of objects into piles is then mirrored in the arranging of marks into groups. In the end, each mark represents an object; its position in one column or the other indicates whether or not that object has a given attribute.

There is no single correct way to represent categorical data-and the Standards do not require Grade 1 students to use any specific format. However, students should be familiar with mark schemes like the one shown in the figure. Another format that might be useful in Grade 1 is a picture graph in which one picture represents one object. (Note that picture graphs are not an expectation in the Standards until Grade 2.) If different students devise different ways to represent the same data set, then the class might discuss relative strengths and weaknesses of each scheme (MP5).

Students' data work in Grade 1 has important connections to addition and subtraction, as noted in Table 1. Students in Grade 1 can ask and answer questions about categorical data based on a representation of the data. For example, with reference to the figure above, a student might ask how many specimens there were altogether, representing this problem by writing an equation such as $7 + 8 = \square$. Students can also ask and answer questions leading to other kinds of addition and subtraction problems (1.OA), such as compare problems or problems involving the addition of three numbers (for situations with three categories).

Grade 2

Students in Grade 2 draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. They solve simple put-together, take-apart, and compare problems using information presented in a bar graph.[2.MD.10, 2.OA.1]

The illustration shows an activity in which students make a bar graph to represent categorical data, then solve addition and subtraction problems based on the data. Students might use scissors to cut out the pictures of each organism and then sort the organisms into piles by category. Category counts might be recorded efficiently in the form of a table.

A bar graph representing categorical data displays no additional information beyond the category counts. In such a graph, the bars are a way to make the category counts easy to interpret visually. Thus, the word problem in part 4 could be solved without drawing a bar graph, just by using the category counts. The problem could even be cast entirely in words, without the accompanying picture: "There are 9 insects, 4 spiders, 13 vertebrates, and 2 organisms of other kinds. How many more spiders would there have to be in order for the number of spiders to equal the number of vertebrates?" Of course, in solving this problem, students would not need to participate in categorizing data or representing it.

Scales in bar graphs Consider the two bar graphs shown to the right, in which the bars are oriented vertically. (Bars in a bar graph can also be oriented horizontally, in which case the following discussion would be modified in the obvious way.) Both of these bar graphs represent the same data set.

These examples illustrate that the horizontal axis in a bar graph of categorical data is not a scale of any kind; position along the horizontal axis has no numerical meaning. Thus, the horizontal position and ordering of the bars are not determined by the data.•

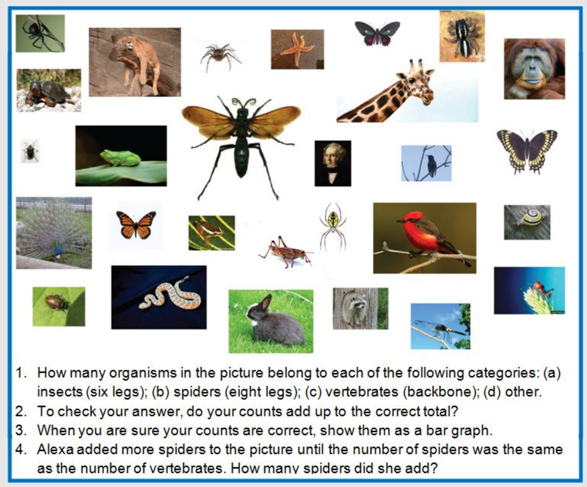
However, the vertical axes in these graphs do have numerical meaning. In fact, the vertical axes in these graphs are segments of number line diagrams. We might think of the vertical axis as a "count scale" (a scale showing counts in whole numbers)—as opposed to a measurement scale, which can be subdivided into fractions of a measurement unit.

Because the count scale in a bar graph is a segment of a number line diagram, when we answer a question such as "How many

2.MD.10 Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

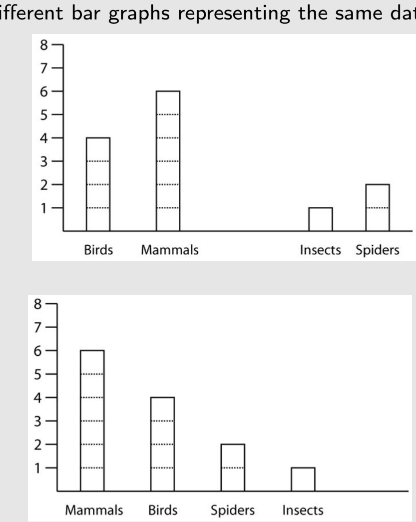
2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Activity for representing categorical data



Students might reflect on the way in which the category counts in part 1 of the activity enable them to efficiently solve the word problem in part 4. (The word problem in part 4 would be difficult to solve directly using just the array of images.)

Different bar graphs representing the same data set



• To minimize potential confusion, it might help to avoid presenting students with examples of categorical data in which the categories are named using numerals, e.g., "Candidate 1," "Candidate 2," "Candidate 3." This will ensure that the only numbers present in the display are found along the count scale.

more birds are there than spiders?" we are finding differences on a number line diagram.[2.MD.6]

When drawing bar graphs on grid paper, the tick marks on the count scale should be drawn at intersections of the gridlines. The tops of the bars should reach the respective gridlines of the appropriate tick marks. When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper.

Students could discuss ways in which bar orientation (horizontal or vertical), order, thickness, spacing, shading, colors, and so forth make the bar graphs easier or more difficult to interpret. By middle school, students could make thoughtful design choices about data displays, rather than just accepting the defaults in a software program (MP5).

Grade 3

In Grade 3, the most important development in data representation for categorical data is that students now draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor in order to yield the number of objects in the given category. These developments connect with the emphasis on multiplication in this grade.

At the end of Grade 3, students can draw a scaled picture graph or a scaled bar graph to represent a data set with several categories (six or fewer categories).[3.MD.3] They can solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.[3.OA.3,3.OA.8] See the examples in the margin, one of which involves categories of shapes.[3.G.1] As in Grade 2, category counts might be recorded efficiently in the form of a table.

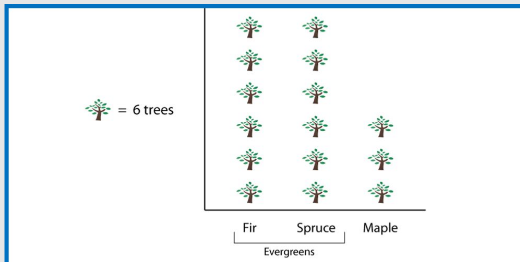
Students can gather categorical data in authentic contexts, including contexts arising in their study of science, history, health, and so on. Of course, students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets. The Standards in Grades 1–3 do not require students to gather categorical data.

Where the Categorical Data Progression is heading

Students' work with categorical data in early grades will develop into later work with bivariate categorical data and two-way tables in eighth grade. "Bivariate categorical data" are data that are categorized according to two attributes. For example, if there is an outbreak of stomach illness on a cruise ship, then passengers might be sorted in two different ways: by determining who got sick and

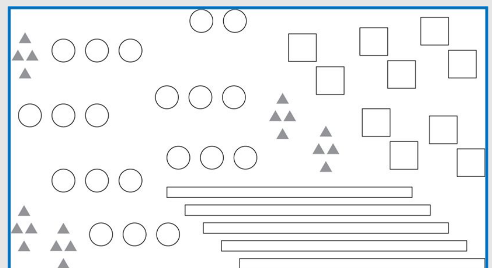
2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A problem about interpreting a scaled picture graph



The picture graph shows how many trees of each kind were in the arboretum. How many more evergreen trees than maple trees are there? (Fir trees and spruce trees are evergreen trees.)

Problems involving categorical data



1. How many shapes above belong to each of the following categories: (a) rectangles; (b) circles; (c) triangles.
2. When you are sure your counts are correct, show them as a picture graph. Use a scale with 1 picture = 5 shapes.
3. Alex added circles to the picture until the number of circles was the same as the number of polygons. How many circles did Alex add?

3.MD.3 Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities,. . .

3.OA.8 Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity . . .

3.G.1 Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals . . .

who didn't, and by determining who ate the shellfish and who didn't. This double categorization—normally shown in the form of a two-way table—might show a strong positive or negative association, in which case it might used to support or contest (but not prove or disprove) a claim about whether the shellfish was the cause of the illness.[8.SP.4]

[8.SP.4] Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.

9

Measurement Data

Grade 2

Students in Grade 2 measure lengths to generate a set of measurement data.[2.MD.1] For example, each student might measure the length of his or her arm in centimeters, or every student might measure the height of a statue in inches. (Students might also generate their own ideas about what to measure.) The resulting data set will be a list of observations, for example as shown in the margin on the following page for the scenario of 28 students each measuring the height of a statue. (This is a larger data set than students would normally be expected to work with in elementary grades.)

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

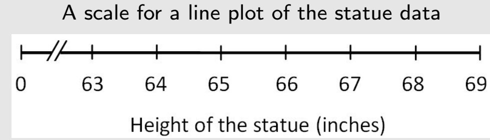
How might one summarize this data set or display it visually? Because students in Grade 2 are already familiar with categorical data and bar graphs, a student might find it natural to summarize this data set by viewing it in terms of categories—the categories in question being the six distinct height values which appear in the data (63 inches, 64 inches, 65 inches, 66 inches, 67 inches, and 69 inches). For example, the student might want to say that there are four observations in the "category" of 67 inches. However, it is important to recognize that 64 inches is not a category like "spiders." Unlike "spiders," 63 inches is a numerical value with a measurement unit. That difference is why the data in this table are called measurement data.

A display of measurement data must present the measured values with their appropriate magnitudes and spacing on the measurement scale in question (length, temperature, liquid capacity, etc.). One method for doing this is to make a *line plot*. This activity connects with other work students are doing in measurement in Grade 2: representing whole numbers on number line diagrams, and representing sums and differences on such diagrams.[2.MD.5,2.MD.6]

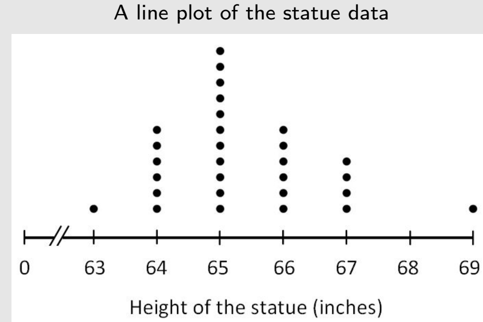
2.MD.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A scale for a line plot of the statue data



A line plot of the statue data



To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: 63 inches and 69 inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale.

Note that the value 68 inches, which was not present in the data set, has been written in proper position midway between 67 inches and 69 inches. (This need to fill in gaps does not exist for a categorical data set; there no "gap" between categories such as fish and spiders!)

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. If a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the observations, or to do any counting of them, before producing the line plot. (In fact, one could even assemble the line plot as the data are being collected, at the expense of having a record of who made what measurement. Students might discuss whether such a record is valuable and why.)

Students might enjoy discussing and interpreting visual features of line plots, such as the "outlier" value of 69 inches in this line plot. (Did student #13 make a serious error in measuring the statue's height? Or in fact is student #13 the only person in the class who measured the height correctly?) However, in Grade 2 the only requirement of the Standards dealing with measurement data is that students generate measurement data and build line plots to display the resulting data sets. (Students do not have to generate the data every time they work on making line plots. That would be too time-consuming. After some experiences in generating the data, most work in producing line plots can be done by providing students with data sets.)

Grid paper might not be as useful for drawing line plots as it is for bar graphs, because the count scale on a line plot is seldom shown for the small data sets encountered in the elementary grades. Additionally, grid paper is usually based on a square grid, but the count scale and the measurement scale of a line plot are conceptually distinct, and there is no need for the measurement unit on the measurement scale to be drawn the same size as the counting unit on the count scale.

Grade 3

In Grade 3, students are beginning to learn fraction concepts (3.NF). They understand fraction equivalence in simple cases, and they use visual fraction models to represent and order fractions. Grade 3 students also measure lengths using rulers marked with halves and fourths of an inch. They use their developing knowledge of fractions and number lines to extend their work from the previous grade by working with measurement data involving fractional measurement values.

For example, every student in the class might measure the height of a bamboo shoot growing in the classroom, leading to the data set shown in the table. (Again, this illustration shows a larger data set than students would normally work with in elementary grades.)

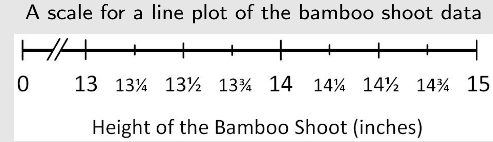
To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: $13\frac{1}{2}$ inches and $14\frac{3}{4}$ inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale. This is just like part of the scale on a ruler.

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. As with Grade 2 line plots, if a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the

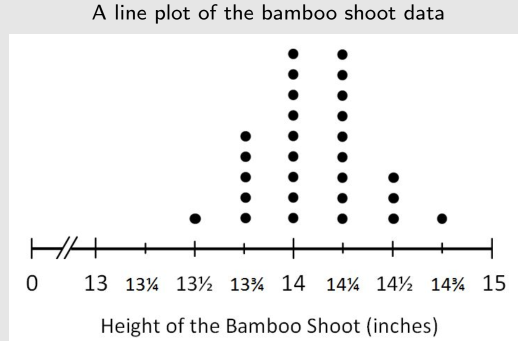
Students' measurements of a statue and of a bamboo shoot

Statue measurements		Bamboo shoot measurements	
Student's initials	Student's measured value (inches)	Student's initials	Height value (inches)
W.B.	64	W.B.	13 3/4
D.W.	65	D.W.	14 1/2
H.D.	65	H.D.	14 1/4
G.W.	65	G.W.	14 3/4
V.Y.	67	V.Y.	14 1/4
T.T.	66	T.T.	14 1/2
D.F.	67	D.F.	14
B.H.	65	B.H.	13 1/2
H.H.	63	H.H.	14 1/4
V.H.	64	V.H.	14 1/4
I.O.	64	I.O.	14 1/4
W.N.	65	W.N.	14
B.P.	69	B.P.	14 1/2
V.A.	65	V.A.	13 3/4
H.L.	66	H.L.	14
O.M.	64	O.M.	13 3/4
L.E.	65	L.E.	14 1/4
M.J.	66	M.J.	13 3/4
T.D.	66	T.D.	14 1/4
K.P.	64	K.P.	14
H.N.	65	H.N.	14
W.M.	67	W.M.	14
C.Z.	64	C.Z.	13 3/4
J.I.	66	J.I.	14
M.S.	66	M.S.	14 1/4
T.C.	65	T.C.	14
G.V.	67	G.V.	14
O.F.	65	O.F.	14 1/4

A scale for a line plot of the bamboo shoot data



A line plot of the bamboo shoot data



observations, or to do any counting of them, before producing the line plot.

Students can pose questions about data presented in line plots, such as how many students obtained measurements larger than $14\frac{1}{4}$ inches.

Grades 4 and 5

Grade 4 students learn elements of fraction equivalence[4.NF.1] and arithmetic, including multiplying a fraction by a whole number[4.NF.4] and adding and subtracting fractions with like denominators.[4.NF.3] Students can use these skills to solve problems, including problems that arise from analyzing line plots. For example, with reference to the line plot above, students might find the difference between the greatest and least values in the data. (In solving such problems, students may need to label the measurement scale in eighths so as to produce like denominators. Decimal data can also be used in this grade.)

Grade 5 students grow in their skill and understanding of fraction arithmetic, including multiplying a fraction by a fraction,[5.NF.4] dividing a unit fraction by a whole number or a whole number by a unit fraction,[4.NF.7] and adding and subtracting fractions with unlike denominators.[5.NF.1] Students can use these skills to solve problems,[5.NF.2,5.NF.6,5.NF.7c] including problems that arise from analyzing line plots. For example, given five graduated cylinders with different measures of liquid in each, students might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. (Students in Grade 6 will view the answer to this question as the mean value for the data set in question.)

As in earlier grades, students should work with data in science and other subjects. Grade 5 students working in these contexts should be able to give deeper interpretations of data than in earlier grades, such as interpretations that involve informal recognition of pronounced differences in populations. This prefigures the work they will do in middle school involving distributions, comparisons of populations, and inference.

Where the Measurement Data Progression is heading

Connection to Statistics and Probability By the end of Grade 5, students should be comfortable making line plots for measurement data and analyzing data shown in the form of a line plot. In Grade 6, students will take an important step toward statistical reasoning per se when they approach line plots as pictures of distributions with features such as clustering and outliers.

Students' work with line plots during the elementary grades develops in two distinct ways during middle school. The first development comes in sixth grade,[6.SP.4] when *histograms* are used.[1] Like

[4.NF.1] Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

[4.NF.3] Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

[5.NF.1] Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

[5.NF.2] Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

[5.NF.6] Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.7c] Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

[6.SP.4] Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

line plots, histograms have a measurement scale and a count scale; thus, a histogram is a natural evolution of a line plot and is used for similar kinds of data (univariate measurement data, the kind of data discussed above).

The other evolution of line plots in middle school is arguably more important. It involves the graphing of bivariate measurement data.[8.SP.1-3] "Bivariate measurement data" are data that represent two measurements. For example, if you take a temperature reading every ten minutes, then every data point is a measurement of temperature as well as a measurement of time. Representing two measurements requires two measurement scales—or in other words, a coordinate plane in which the two axes are each marked in the relevant measurement units. Representations of bivariate measurement data in the coordinate plane are called *scatter plots*. In the case where one axis is a time scale, they are called *time graphs* or *line graphs*. Time graphs can be used to visualize trends over time, and scatter plots can be used to discover associations between measured variables in general.

8.SP.1 Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

8.SP.2 Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

8.SP.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

Connection to the Number System The Standards do not explicitly require students to create time graphs. However, it might be considered valuable to expose students to time series data and to time graphs as part of their work in meeting standard 6.NS.8. For example, students could create time graphs of temperature measured each hour over a 24-hour period, where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day. It is traditional to connect ordered pairs with line segments in such a graph, not in order to make any claims about the actual temperature value at unmeasured times, but simply to aid the eye in perceiving trends.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

[1] To display a set of measurement data with a histogram, specify a set of nonoverlapping intervals along the measurement scale. Then, instead of showing each individual measurement as a dot, use a bar oriented along the count scale to indicate the number of measurements lying within each interval on the measurement scale. A histogram is thus a little like a bar graph for categorical data, except that the "categories" are successive intervals along a measurement scale. (Note that the Standards follow the GAISE report in reserving the term "categorical data" for non-numerical categories. In the Standards, as in GAISE (see p. 35), bar graphs are for categorical data with non-numerical categories, while histograms are for measurement data which has been grouped by intervals along the measurement scale.)

13

Appendix: Additional Examples

These examples show some rich possibilities for data work in K–8. The examples are not shown by grade level because each includes some aspects that go beyond the expectations stated in the Standards.

Example 1. Comparing bar graphs

Are younger students lighter sleepers than older students? To study this question a class first agreed on definitions for light, medium and heavy sleepers and then collected data from first and fifth grade students on their sleeping habits. The results are shown in the margin.

How do the patterns differ? What is the typical value for first graders? What is the typical value for fifth graders? Which of these groups appears to be the heavier sleepers?

Example 2. Comparing line plots

Fourth grade students interested in seeing how heights of students change for kids around their age measured the heights of a sample of eight-year-olds and a sample of ten-year-olds. Their data are plotted in the margin.

Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?

Example 3. Fair share averaging

Ten students decide to have a pizza party and each is asked to bring his or her favorite pizza. The amount paid (in dollars) for each pizza is shown in the plot to the right.

Each of the ten is asked to contribute an equal amount (his or her fair share) to the cost of the pizza. Where does that fair share amount lie on the plot? Is it closer to the smaller values or the large one? Now, two more students show up for the party and they have contributed no pizza. Plot their values on the graph and calculate a new fair share. Where does it lie on the plot? How many more students without pizza would have to show up to bring the fair share cost below $8.00?